AN

INTRODUCTION

TO THE

ELEMENTS OF ALGEBRA,

DESIGNED FOR THE USE OF THOSE

WHO ARE ACQUAINTED ONLY WITH THE FIRST PRINCIPLES

Euler, Leonard

OF

ARITHMETIC.

SELECTED FROM THE ALGEBRA OF EULER.

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ADVERTISEMENT.

NONE but those who are just entering upon the study of Mathematics need to be informed of the high character of Euler's Algebra. It has been allowed to hold the very first place among elementary works upon this subject. The author was a man of genius. He did not, like most writers, compile from others. He wrote from his own reflections. He simplified and improved what was known, and added much that was new. He is particularly distinguished for the clearness and comprehensiveness of his views. He seems to have the subject of which he treats present to his mind in all its relations and bearings before he begins to write. The parts of it are arranged in the most admirable order. Each step is introduced by the preceding, and leads to that which follows, and the whole taken together constitutes an entire and connected piece, like a highly wrought story.

This author is remarkable also for his illustrations. He teaches by instances. He presents one example after another, each evident by itself, and each throwing some new light upon the subject, till the reader begins to anticipate for himself the truth to be inculcated,

Some opinion may be formed of the adaptation of this treatise to learners, from the circumstances under which it was composed. It was undertaken after the author became blind, and was dictated to a young man entirely without education, who by this means became an expert algebraist, and was able to render the author important services as an amanuensis. It was written originally in German. It has since been translated into Russian, French, and English, with notes and additions.

The entire work consists of two volumes octavo, and contains many things intended for the professed mathematician, rather than the general student. It was thought that a selection of such parts as would form an easy introduction to the science would be well received, and tend to promote a taste for analysis among students, and to raise the character of mathematical learning.

Notwithstanding the high estimation in which this work has been held, it is scarcely to be met with in the country, and is very little known in England. On the continent of Europe this author is the constant theme of eulogy. His writings have the character of classics. They are regarded at the same time as the most profound and the most perspicuous, and as affording the finest models of analysis. They furnish the germs of the most approved elementary works on the different branches of this science. The constant reply of one of the first mathematicians* of France to those who consulted him upon the best method of studying mathematics was, "*study Euler.*" "It is needless," said he, "to accumulate books; true lovers of mathematics will always read Euler; because in his writings every thing is clear, distinct, and correct; because they swarm with excellent examples; and because it is always necessary to have recourse to the fountain head."

The selections here offered are from the first English edition. A few errors have been corrected and a few alterations made in the phraseology. In the original no

*Lagrange.

questions were left to be performed by the learner. A collection was made by the English translator, and subjoined at the end, with references to the sections to which they relate. These have been mostly retained, and some new ones have been added.

Although this work is intended particularly for the algebraical student, it will be found to contain a clear and full explanation of the fundamental principles of arithmetic; vulgar fractions, the doctrine of roots and powers, of the different kinds of proportion and progression, are treated in a manner that can hardly fail to interest the learner and make him acquainted with the reason of those rules which he has so frequent occasion to apply.

JOHN FARRAR,
Professor of Mathematics and Natural Philosophy in the University at Cambridge.

Cambridge, February, 1818.

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INTRODUCTION

TO THE

ELEMENTS OF ALGEBRA.

SECTION I.

OF THE DIFFERENT METHODS OF CALCULATION APPLIED TO SIMPLE QUANTITIES.

CHAPTER I.

Of Mathematics in general.

ARTICLE 1. Whatever is capable of increase or diminution, is called *magnitude* or *quantity.*

A sum of money, for instance, is a quantity, since we may increase it or diminish it. The same may be said with respect to any given weight, and other things of this nature.

2. From this definition it is evident, that there must be so many different kinds of magnitude as to render it difficult even to enumerate them; and this is the origin of the different branches of mathematics, each being employed on a particular kind of magnitude. Mathematics, in general, is the *science of quantity;* or the science which investigates the means of measuring quantity.

3. Now we cannot measure or determine any quantity, except by considering some other quantity of the same kind as known, and pointing out their mutual relation. If it were proposed, for example, to determine the quantity of a sum of money, we should take some known piece of money (as a dollar, a crown, a ducat, or some other coin), and show how many of these pieces are contained in the

given sum. In the same manner, if it were proposed to determine the quantity of a weight, we should take a certain known weight; for example, a pound, an ounce, &c., and then show how many times one of these weights is contained in that which we are endeavoring to ascertain. If we wished to measure any length or extension, we should make use of some known length, as a foot for example.

4. So that the determination, or the measure of magnitude of all kinds, is reduced to this: fix at pleasure upon any one known magnitude of the same species with that which is to be determined, and consider it as the *measure* or *unit*; then, determine the proportion of the proposed magnitude to this known measure. This proportion is always expressed by numbers; so that a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit.

5. From this it appears, that all magnitudes may be expressed by numbers; and that the foundation of all the mathematical sciences must be laid in a complete treatise on the science of numbers; and in an accurate examination of the different possible methods of calculation.

This fundamental part of mathematics is called *analysis*, or *algebra*.

6. In algebra then we consider only numbers which represent quantities, without regarding the different kinds of quantity. These are the subjects of other branches of the mathematics.

7. Arithmetic treats of numbers in particular, and is the *science of numbers properly so called*; but this science extends only to certain methods of calculation which occur in common practice: algebra, on the contrary, comprehends in general all the cases which can exist in the doctrine and calculation of numbers.

CHAPTER II.

Explanation of the signs + plus *and* — minus.

8. When we have to add one given number to another, this is indicated by the sign + which is placed before the second number, and is read *plus*. Thus 5 + 3 signifies that we must add 3 to the number 5, and every one knows that the result is 8; in the same manner 12 + 7 make 19; 25 + 16 make 41; the sum of 25 + 41 is 66, &c.

9. We also make use of the same sign + or *plus*, to connect several members together; for example, 7 + 5 + 9 signifies that to the number 7 we must add 5 and also 9, which make 21. The reader will therefore understand what is meant by

$$8 + 5 + 13 + 11 + 1 + 3 + 10;$$

viz. the sum of all these numbers, which is 51.

10. All this is evident; and we have only to mention, that, in algebra, in order to generalize numbers, we represent them by letters, as a, b, c, d, &c. Thus the expression $a + b$ signifies the sum of two numbers, which we express by a and b, and these numbers may be either very great or very small. In the same manner, $f + m + b + x$, signifies the sum of the numbers represented by these four letters.

If we know, therefore, the numbers that are represented by letters, we shall at all times be able to find, by arithmetic, the sum or value of similar expressions.

11. When it is required, on the contrary, to subtract one given number from another, this operation is denoted by the sign —, which signifies *minus*, and is placed before the number to be subtracted: thus 8 — 5 signifies that the number 5 is to be taken from the number 8; which being done, there remains 3. In like manner 12 — 7 is the same as 5; and 20 — 14 is the same as 6, &c.

12. Sometimes also we may have several numbers to be subtracted from a single one; as for instance,

$$50 - 1 - 3 - 5 - 7 - 9.$$

This signifies, first, take 1 from 50, there remains 49; take 3 from that remainder, there will remain 46; take away 5, 41 remains; take away 7, 34 remains; lastly, from that take 9, and there remains 25; this last remainder is the value of the expression. But as the numbers 1, 3, 5, 7, 9, are all to be subtracted, it is the same thing if we subtract their sum, which is 25, at once from 50, and the remainder will be 25 as before.

13. It is also very easy to determine the value of similar expressions, in which both the signs + *plus* and — *minus* are found: for example;

$$12 - 3 - 5 + 2 - 1 \text{ is the same as } 5.$$

We have only to collect separately the sum of the numbers that have the sign + before them, and subtract from it the sum of those that have the sign —. The sum of 12 and 2 is 14; that of 3, 5, and 1, is 9; now 9 being taken from 14, there remains 5.

14. It will be perceived from these examples that *the order in which we write the numbers is quite indifferent and arbitrary, provided the proper sign of each be preserved.* We might with equal propriety have arranged the expression in the preceding article thus; $12 + 2 - 5 - 3 - 1$, or $2 - 1 - 3 - 5 + 12$, or $2 + 12 - 3 - 1 - 5$, or in still different orders. It must be observed, that in the expression proposed, the sign + is supposed to be placed before the number 12.

15. It will not be attended with any more difficulty, if, in order to generalize these operations, we make use of letters instead of real numbers. It is evident, for example, that

$$a - b - c - + d - e$$

signifies that we have numbers expressed by a and d, and that from these numbers, or from their sum, we must subtract the numbers expressed by the letters b, c, e, which have before them the sign —.

16. Hence it is absolutely necessary to consider what sign is prefixed to each number: for *in algebra, simple quantities are numbers considered with regard to the signs which precede, or affect them.* Further, we call those *positive quantities,* before which the sign + is found; and those are called *negative quantities,* which are affected with the sign —.

17. The manner in which we generally calculate a person's property, is a good illustration of what has just been said. We denote what a man really possesses by positive numbers, using, or understanding the sign +; whereas his debts are represented by

negative numbers, or by using the sign —. Thus, when it is said of any one that he has 100 crowns, but owes 50, this means that his property really amounts to 100 — 50; or, which is the same thing, + 100 — 50, that is to say 50.

18. As negative numbers may be considered as debts, because positive numbers represent real possessions, we may say that negative numbers are less than nothing. Thus, when a man has nothing in the world, and even owes 50 crowns, it is certain that he has 50 crowns less than nothing; for if any one were to make him a present of 50 crowns to pay his debts, he would still be only at the point nothing, though really richer than before.

19. In the same manner, therefore, as positive numbers are incontestably greater than nothing, negative numbers are less than nothing.* Now we obtain positive numbers by adding 1 to 0, that is to say, to nothing; and by continuing always to increase thus from unity. This is the origin of the series of numbers called *natural numbers;* the following are the leading terms of this series:

$$0, +1, +2, +3, +4, +5, +6, +7, +8, +9, +10,$$

and so on to infinity.

But if instead of continuing this series by successive additions, we continued it in the opposite direction, by perpetually subtracting unity, we should have the series of negative numbers:

$$0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10,$$

and so on to infinity.

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* By being less than nothing is meant simply, that they are of such a nature as to cancel or destroy an equal number with the sign *plus* before it, so that — 4, or — a is as really a positive thing, and is as easily conceived, as + 4 or + a. The quantity 4 or a may be considered independently of its sign. The sign + implies that this quantity is to be added, and the sign — that it is to be subtracted. This subject may be illustrated by the scale of a thermometer. After observing the mercury to stand at 50°, for instance, if I am told, that it has changed 4°, I have a distinct idea of the portion of the scale denoted by four of its divisions, without applying them in any particular direction. But when I am further informed that this change of the thermometer is — or *subtractive* with respect to its former state, I then understand that the mercury stands at 46°, whereas it would be at 54° if the change had been + or additive.

20. All these numbers, whether positive or negative, have the known appellation of whole numbers, or *integers*, which consequently are either greater or less than nothing. We call them *integers*, to distinguish them from fractions, and from several other kinds of numbers of which we shall hereafter speak. For instance, 50 being greater by an entire unit than 49, it is easy to comprehend that there may be between 49 and 50 an infinity of intermediate numbers, all greater than 49, and yet all less than 50. We need only imagine two lines, one 50 feet, the other 49 feet long, and it is evident that there may be drawn an infinite number of lines all longer than 49 feet, and yet shorter than 50.

21. It is of the utmost importance, through the whole of algebra, that a precise idea be formed of those negative quantities about which we have been speaking. I shall content myself with remarking here that all such expressions, as

$$+1-1,\ +2-2,\ +3-3,\ +4-4, \text{ \&c.}$$

are equal to 0 or nothing. And that

$$+2-5 \text{ is equal to } -3.$$

For if a person has 2 crowns, and owes 5, he has not only nothing, but still owes 3 crowns: in the same manner,

$$7-12 \text{ is equal to } -5, \text{ and } 25-40 \text{ is equal to } -15.$$

22. The same observations hold true, when to make the expression more general, letters are used instead of numbers: 0 or nothing will always be the value of $+a-a$. If we wish to know the value $+a-b$ two cases are to be considered.

The first is when a is greater than b; b must then be subtracted from a, and the remainder (before which is placed or understood to be placed the sign $+$) shows the value sought.

The second case is that in which a is less than b; here a is to be subtracted from b, and the remainder being made negative, by placing before it the sign $-$, will be the value sought.

CHAPTER III.

Of the Multiplication of Simple Quantities.

23. When there are two or more equal numbers to be added together, the expression of their sum may be abridged; for example,

$a + a$ is the same with $2 \times a$,

$a + a + a$ $3 \times a$,

$a + a + a + a$ $4 \times a$, and so on; where $\times$ is the sign of multiplication. In this manner we may form an idea of multiplication; and it is to be observed that,

$2 \times a$ signifies 2 times, or twice a

$3 \times a$ 3 times, or thrice a

$4 \times a$ 4 times a, &c.

24. *If therefore a number expressed by a letter is to be multiplied by any other number, we simply put that number before the letter;* thus,

a multiplied by 20 is expressed by $20\,a$, and

b multiplied by 30 gives $30\,b$, &c.

It is evident also that c taken once, or $1\,c$, is just c.

25. Further it is extremely easy to multiply such products again by other numbers; for example:

2 times, or twice $3\,a$ makes $6\,a$,

3 times, or thrice $4\,b$ makes $12\,b$,

5 times $7\,x$ makes $35\,x$,

and these products may be still multiplied by other numbers at pleasure.

26. *When the number, by which we are to multiply, is also represented by a letter, we place it immediately before the other letter;* thus, in mutiplying b by a, the product is written $a\,b$; and $p\,q$ will be the product of the multiplication of the number q by p. If we multiply this $p\,q$ again by a, we shall obtain $a\,p\,q$.

27. It may be remarked here, that *the order in which the letters are joined together is indifferent;* that $a\,b$ is the same thing as $b\,a$; for b multiplied by a produces as much as a multiplied by b. To understand this, we have only to substitute for a and b known numbers, as 3 and 4; and the truth will be self-evident; for 3 times 4 is the same as 4 times 3.

28. It will not be difficult to perceive, that when you have to put numbers in the place of letters joined together, as we have described, they cannot be written in the same manner by putting them one after the other. For if we were to write 34 for 3 times 4, we should have 34 and not 12. When, therefore, it is required to multiply common numbers, we must separate them by the sign $\times$, or points: thus, 3×4, or $3 . 4$, signifies 3 times 4, that is 12. So, 1×2 is equal to 2; and $1 \times 2 \times 3$ makes 6. In like manner $1 \times 2 \times 3 \times 4 \times 56$ makes 1344; and $1 \times 2 \times 2 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$ is equal to 3628800, &c.

29. In the same manner, we may discover the value of an expression of this form, $5 . 7 . 8\, a\, b\, c\, d$. It shows that 5 must be multiplied by 7, and that this product is to be again multiplied by 8; that we are then to multiply this product of the three numbers by a, next by b, and then by c, and lastly by d. It may be observed also, that instead of $5 \times 7 \times 8$ we may write its value, 280; for we obtain this number when we multiply the product of 5 by 7, or 35, by 8.

30. The results which arise from the multiplication of two or more numbers are called *products*; and the numbers, or individual letters, are called *factors*.

31. Hitherto we have considered only positive numbers, and there can be no doubt, but that the products which we have seen arise are positive also: *viz.* $+ a$ by $+ b$ must necessarily give $+ a b$. But we must separately examine what the multiplication of $+ a$ by $- b$, and of $- a$ by $- b$, will produce.

32. Let us begin by multiplying $- a$ by 3 or $+ 3$; now since $- a$ may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is $- 3 a$. So if we multiply $- a$ by $+ b$, we shall obtain $- b a$, or which is the same thing, $- a b$. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative; and lay it down as a rule, that $+$ by $+$ makes $+$, or *plus*, and that on the contrary $+$ by $-$, or $-$ by $+$ gives $-$, or *minus*.

33. It remains to resolve the case in which $-$ is multiplied by $-$; or, for example, $- a$ by $- b$. It is evident, at first sight, with regard to the letters, that the product will be $a b$; but it is doubtful whether the sign $+$, or the sign $-$, is to be placed before the product; all we know is, that it must be one or the other of

these signs. Now I say, that it cannot be the sign —: for — a by + b gives — a b, and — a by — b cannot produce the same result as — a by + b; but must produce a contrary result, that is to say, + a b; consequently we have the following rule: — multiplied by — produces +, in the same manner as + multiplied by +.*

* It is a subject of great embarrassment and perplexity to learners to conceive how the product of two negative quantities should be positive. This arises from the idea they receive of the nature of multiplication as explained and applied in arithmetic, where positive quantities only are employed. The term is used in a more enlarged sense when negative quantities are concerned, as may be shown without making use of letters. If I wished to multiply, for instance, 9 — 5 (or 9 diminished by 5) by 3, I should first find the product of 9 by 3, or 27. But this is evidently taking the multiplicand too great by 5, and of course the product too great by 3 times 5; I accordingly write for the product 27 — 15, equivalent to 12, which is the product that would arise from first performing the subtraction indicated by the sign —, and using the result as the multiplicand. Thus,

Multiplicand	9 — 5	which is equal to	4
Multiplier	3		3
Product	27 — 15	which is equal to	12

Let us now take for the multiplier the quantity 7 — 4, which is equivalent to 3. We multiply, in the first place by 7, in the manner that we have just done by 3, and the result is 63 — 35. But as the multiplier is 7 diminished by 4, multiplying by 7 must give 4 times too much. Accordingly we take 4 times the multiplicand, or 36 — 20 and subtract this from 63 — 35, or 7 times the multiplicand. Now in making this subtraction it is to be observed that the subtrahend 36 — 20 is 36 diminished by 20, and if we subtract 36 we take away too much by 20, and must therefore add this latter quantity. Consequently the true product will be 63 — 35 — 36 + 20, equivalent to 12, as before. Thus this mode of proceeding gives the same result as that obtained by first performing the subtractions indicated in the latter term of the multiplicand and multiplier. The several steps in each case are as follows:

Multiplicand	9 — 5	which is equal to	4
Multiplier	7 — 4	which is equal to	3
	63 — 35	Product	12
	— 36 + 20		

Product 63 — 35 — 36 + 20 or 83 — 71, that is, 12.

34. The rules which we have explained are expressed more briefly as follows:

Like signs multiplied together, give + ; *unlike or contrary signs give* —. Thus, when it is required to multiply the following num-

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Thus we see that 7 or + 7 by — 5 gives — 35, and — 4 by + 9 gives — 36, and — 4 by — 5 gives + 20. The same general reasoning will apply when letters are used instead of numbers.

$$\begin{array}{lr} \text{Multiplicand} & a - b \\ \text{Multiplier} & c - d \\ \hline & ac - bc \\ & - ad + bd \\ \hline \text{Product} & ac - bc - ad + bd. \end{array}$$

We say in this case, that when we multiply a by c we take the multiplicand too great by b, and must therefore diminish the result $a\,c$ by the product of b by c or $b\,c$. So also in multiplying the two terms of the multiplicand by c, we have taken the multiplier too great by d, and must therefore diminish the result $a\,c - b\,c$ by the product of $a - b$ by d, or $a\,d - b\,d$. But if we subtract the whole of $a\,d$, we subtract too much by $b\,d$; $b\,d$ must accordingly be added.

The rule for negative quantities here illustrated is not necessary where mere numbers are employed, because the subtraction indicated may always be performed. But this cannot be done with respect to letters which stand for no particular values, but are intended as general expressions of quantities.

The truth of the rule may be shown also when applied to quantities taken singly. We say that multiplying one quantity by another is taking one as many times as there are units in the other, and the result is the same, whichever of the quantities be taken for the multiplicand. Thus multiplying 9 by 3 is taking 9 three times, or which is the same thing, taking 3 nine times (Arith. 27). But in arithmetic, quantities are always taken affirmatively, that is additively. When, therefore, we take 9 or + 9 three times additively, or + 3 nine times additively, the result will evidently be additive or + 27. When, on the contrary, one of the factors is negative, as for instance, in multiplying — 5 by + 3; in this case, — 5 is to be taken three times additively, and — 5 added to — 5 added to — 5 is clearly — 15. So also if we consider + 3 as the multiplicand, then + 3 is to be taken five times subtractively; now 3 taken subtractively once (or which is the same thing, 3 × — 1) is equivalent to — 3, taken bers; $+ a, - b, - c, + d$; we have first $+ a$ multiplied by $- b$, which makes $- a b$; this by $- c$, gives $+ a b c$; and this by $+ d$, gives $+ a b c d$.

35. The difficulties with respect to the signs being removed, we have only to show how to multiply numbers that are themselves products. If we were, for instance, to multiply the number $a b$ by the number $c d$, the product would be $a b c d$, and it is obtained by multiplying first $a b$ by c, and then the result of that multiplication by d. Or, if we had to multiply 36 by 12; since 12 is equal to 3 times 4, we should only multiply 36 first by 3, and then the product 108 by 4, in order to have the whole product of the multiplication of 12 by 36, which is consequently 432.

36. But if we wished to multiply $5 a b$ by $3 c d$, we might write $3 c d \times 5 a b$; however, as in the present instance the order of the numbers to be multiplied is indifferent, it will be better, as is also the custom, to place the common numbers before the letters, and to express the product thus: $5 \times 3 a b c d$, or $15 a b c d$; since 5 times 3 is 15.

So if we had to multiply $12 p q r$ by $7 x y$, we should obtain $12 \times 7 p q r x y$, or $84 p q r x y$.

CHAPTER IV.

Of the Nature of Whole Numbers or Integers, with respect to their Factors.

37. We have observed that a product is generated by the multiplication of two or more numbers together, and that these numbers are called *factors*. Thus the numbers a, b, c, d, are the factors of the product $a b c d$.

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subtractively twice is $- 6$, three times is $- 9$, five times is $- 15$. But, when the multiplicand and multiplier are both negative, as in the case of multiplying $- 5$ by $- 4$: here a subtractive quantity is to be taken subtractively, that is, we are to take away successively a diminishing or lessening quantity, which is certainly equivalent to adding an increasing quantity. Thus, if we take away $- 5$ once, we augment the sum with which it is to be connected by $+ 5$; if we take away $- 5$ twice, we make the augmentation $+ 10$; if four times, $+ 20$; that is, $- 5 \times - 4$ is equivalent to $+ 20$.

38. If, therefore, we consider all whole numbers as products of two or more numbers multiplied together, we shall soon find that some cannot result from such a multiplication, and consequently have not any factors; while others may be the products of two or more multiplied together, and may consequently have two or more factors. Thus, 4 is produced by 2×2; 6 by 2×3; 8 by $2 \times 2 \times 2$; or 27 by $3 \times 3 \times 3$; and 10 by 2×5, &c.

39. But, on the other hand, the numbers, 2, 3, 5, 7, 11, 13, 17, &c., cannot be represented in the same manner by factors, unless for that purpose we make use of unity, and represent 2, for instance, by 1×2. Now the numbers which are multiplied by 1, remaining the same, it is not proper to reckon unity as a factor.

All numbers, therefore, such as 2, 3, 5, 7, 11, 13, 17, &c. which cannot be represented by factors, are called *simple*, or *prime numbers;* whereas others, as 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, &c. which may be represented by factors, are called *compound numbers.*

40. *Simple* or *prime numbers* deserve therefore particular attention, since they do not result from the multiplication of two or more numbers. It is particularly worthy of observation that if we write these numbers in succession as they follow each other, thus;

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, &c.

we can trace no regular order; their increments are sometimes greater, sometimes less; and hitherto no one has been able to discover whether they follow any certain law or not.

41. *All compound numbers, which may be represented by factors, result from the prime numbers above mentioned; that is to say, all their factors are prime numbers.* For, if we find a factor which is not a prime number, it may always be decomposed and represented by two or more prime numbers. When we have represented, for instance, the number 30 by 5×6, it is evident that 6 not being a prime number, but being produced by 2×3, we might have represented 30 by $5 \times 2 \times 3$, or by $2 \times 3 \times 5$; that is to say, by factors, which are all prime numbers.

42. If we now consider those compound numbers which may be resolved into prime numbers, we shall observe a great difference among them; we shall find that some have only two factors, that others have three, and others a still greater number. We have already seen, for example, that

4 is the same as	2×2,	6 is the same as	2×3,
8	$2 \times 2 \times 2$,	9	3×3,
10	2×5,	12	$2 \times 3 \times 2$,
14	2×7,	15	3×5,
16	$2 \times 2 \times 2 \times 2$,	and so on.	

43. Hence it is easy to find a method for analysing any number, or resolving it into its simple factors. Let there be proposed, for instance the number 360; we shall represent it first by 2×180. Now 180 is equal to 2×90, and

$$\left.\begin{matrix}90\\45\\15\end{matrix}\right\}\text{ is the same as }\left\{\begin{matrix}2 \times 45,\\3 \times 15,\text{ and lastly}\\3 \times 5.\end{matrix}\right.$$

So that the number 360 may be represented by these simple factors, $2 \times 2 \times 2 \times 3 \times 3 \times 5$; since all these numbers multiplied together produce 360.

44. This shows, that the prime numbers cannot be divided by other numbers, and on the other hand, that *the simple factors of compound numbers are found, most conveniently, and with the greatest certainty, by seeking the simple, or prime numbers, by which those compound numbers are divisible.* But for this, *division* is necessary; we shall therefore explain the rules of that operation in the following chapter.

CHAPTER V.

Of the Division of Simple Quantities.

45. When a number is to be separated into two, three, or more equal parts, it is done by means of *division*, which enables us to determine the magnitude of one of those parts. When we wish, for example, to separate the number 12 into three equal parts, we find by division that each of those parts is equal to 4.

The following terms are made use of in this operation. The number, which is to be decompounded or divided, is called the *dividend*; the number of equal parts sought is called the *divisor;* the magnitude of one of those parts, determined by the division, is called the *quotient*; thus, in the above example;

12 is the dividend,
3 is the divisor, and
4 is the quotient.

46. It follows from this, that if we divide a number by 2, or into two equal parts, one of those parts, or the quotient, taken twice, makes exactly the number proposed ; and, in the same manner, if we have a number to be divided by 3, the quotient taken thrice must give the same number again. In general, *the multiplication of the quotient by the divisor must always reproduce the dividend.*

47. It is for this reason that division is called a rule, which teaches us to find a number or quotient, which, being multiplied by the divisor, will exactly produce the dividend. For example, if 35 is to be divided by 5, we seek a number which, multiplied by 5, will produce 35. Now this number is 7, since 5 times 7 is 35. The mode of expression, employed in this reasoning, is; 5 in 35, 7 times; and 5 times 7 makes 35.

48. The dividend, therefore, may be considered as a product, of which one of the factors is the divisor, and the other the quotient. Thus, supposing we have 63 to divide by 7, we endeavor to find such a product, that taking 7 for one of its factors, the other factor multiplied by this may exactly give 63. Now 7×9 is such a product, and consequently 9 is the quotient obtained when we divide 63 by 7.

49. In general, if we have to divide a number $a\,b$ by a, it is evident that the quotient will be b; for a multiplied by b gives the dividend $a\,b$. It is clear also, that if we had to divide $a\,b$ by b, the quotient would be a. And in all examples of division that can be proposed, if we divide the dividend by the quotient, we shall again obtain the divisor; for as 24 divided by 4 gives 6, so 24 divided by 6 will give 4.

50. As *the whole operation consists in representing the dividend by two factors, of which one shall be equal to the divisor, the other to the quotient;* the following examples will be easily understood. I say first, that the dividend $a\,b\,c$, divided by a, gives $b\,c$; for a, multiplied by $b\,c$, produces $a\,b\,c$: in the same manner $a\,b\,c$ being divided by b, we shall have $a\,c$; and $a\,b\,c$, divided by $a\,c$, gives b. I say also, that $12\,m\,n$, divided by $3\,m$, gives $4\,n$; for $3\,m$, multiplied by $4\,n$ makes $12\,m\cdot n$. But if this same number $12\,m\,n$ had been divided by 12, we should have obtained the quotient $m\,n$.

51. Since every number a may be expressed by $1\,a$ or *one* a, it is evident that if we had to divide a or $1\,a$ by 1, the quotient would be the same number a. But, on the contrary, if the same number a, or $1\,a$, is to be divided by a, the quotient will be 1.

52. It often happens that we cannot represent the dividend as the product of two factors, of which one is equal to the divisor; and then the division cannot be performed in the manner we have described.

When we have, for example, 24 to be divided by 7, it is at first sight obvious, that the number 7 is not a factor of 24; for the product of 7 × 3 is only 21, and consequently too small, and 7 × 4 makes 28, which is greater than 24. We discover, however, from this, that the quotient must be greater than 3, and less than 4. In order, therefore, to determine it exactly, we employ another species of numbers, which are called *fractions*, and which we shall consider in one of the following chapters.

53. Until the use of fractions is considered, it is usual to rest satisfied with the whole number which approaches nearest to the true quotient, but at the same time paying attention to the *remainder* which is left; thus we say, 7 in 24, 3 times, and the remainder is 3, because 3 times 7 produces only 21, which is 3 less than 24. We may consider the following examples in the same manner:

6)34(5, that is to say, the divisor is 6, the dividend 34,
30 the quotient 5, and the remainder 4.

4

9)41(4 here the divisor is 9, the dividend 41, the quotient
364, and the remainder 5.

5

The following rule is to be observed in examples where there is a remainder.

54. *If we multiply the divisor by the quotient, and to the product add the remainder, we must obtain the dividend;* this is the method of proving division, and of discovering whether the calculation is right or not. Thus, in the former of the two last examples, if we multiply 6 by 5, and to the product 30 add the remainder 4, we obtain 34, or the dividend. And in the last example, if we multiply the divisor 9 by the quotient 4, and to the product 36 add the remainder 5, we obtain the dividend 41.

55. Lastly, it is necessary to remark here, with regard to the signs + *plus* and — *minus*, that if we divide $+ ab$ by $+ a$, the quotient will be $+ b$, which is evident. But if we divide $+ ab$ by $- a$ the quotient will be $- b$; because $- a \times - b$ gives $+ ab$.

If the dividend is $-ab$, and is to be divided by the divisor $+a$, the quotient will be $-b$; because it is $-b$, which, multiplied by $+a$, makes $-ab$. Lastly, if we have to divide the dividend $-ab$ by the divisor $-a$, the quotient will be $+b$; for the dividend $-ab$ is the product of $-a$ by $+b$.

56. *With regard, therefore, to the signs + and —, division admits the same rules that we have seen applied in multiplication*, viz.

$+$ by $+$ requires $+$; $+$ by $-$ requires $-$;
$-$ by $+$ requires $-$; $-$ by $-$ requires $+$;

or in a few words, *like signs give plus, unlike signs give minus.*

57. Thus, when we divide $18pq$ by $-3p$, the quotient is $-6q$. Further;

$-30xy$, divided by $+6y$, gives $-5x$, and
$-54abc$, divided by $-9b$, gives $+6ac$;

for in this last example, $-9b$, multiplied by $+6ac$, makes $-6 \times 9abc$, or $-54abc$. But we have said enough on the division of simple quantities; we shall therefore hasten to the explanation of fractions, after having added some farther remarks on the nature of numbers, with respect to their divisors.

CHAPTER VI.

Of the Properties of Integers with respect to their Divisors.

58. As we have seen that some numbers are divisible by certain divisors, while others are not; in order that we may obtain a more particular knowledge of numbers, this difference must be carefully observed, both by distinguishing the numbers that are divisible by divisors from those which are not, and by considering the remainder that is left in the division of the latter. For this purpose let us examine the divisors;

2, 3, 4, 5, 6, 7, 8, 9, 10, &c.

59. First, let the divisor be 2; the numbers divisible by it are 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, &c. which, it appears, increase always by two. These numbers, as far as they can be continued, are called *even numbers*. But there are other numbers, namely,

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, &c.,

which are uniformly less or greater than the former by unity, and which cannot be divided by 2, without the remainder 1; these are called *odd numbers.*

The even numbers are all comprehended in the general expression $2a$; for they are all obtained by successively substituting for a the integers, 1, 2, 3, 4, 5, 6, 7, &c., and hence it follows that the odd numbers are all comprehended in the expression $2a + 1$, because $2a + 1$ is greater by unity than the even number $2a$.

60. In the second place, let the number 3 be the divisor, the numbers divisible by it are,

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, and so on;

and these numbers may be represented by the expression $3a$; for $3a$ divided by 3 gives the quotient a without a remainder. All other numbers, which we would divide by 3, will give 1 or 2 for a remainder, and are consequently of two kinds. Those which, after the division leave the remainder 1, are;

1, 4, 7, 10, 13, 16, 19, &c.,

and are contained in the expression $3a + 1$; but the other kind, where the numbers give the remainder 2, are;

2, 5, 8, 11, 14, 17, 20, &c.,

and they may be generally expressed by $3a + 2$; so that all numbers may be expressed either by $3a$, or by $3a + 1$, or by $3a + 2$.

61. Let us now suppose that 4 is the divisor under consideration; the numbers which it divides are;

4, 8, 12, 16, 20, 24, &c.,

which increase uniformly by 4, and are comprehended in the expression $4a$. All other numbers, that is, those which are not divisible by 4, may leave the remainder 1, or be greater than the former by 1; as

1, 5, 9, 13, 17, 21, 25, &c.,

and consequently may be comprehended in the expression $4a + 1$: or they may give the remainder 2; as

2, 6, 10, 14, 18, 22, 26, &c.,

and be expressed by $4a + 2$; or, lastly, they may give the remainder 3; as

3, 7, 11, 15, 19, 23, 27, &c.,

and may be represented by the expression $4a + 3$.

All possible integral numbers are therefore contained in one or other of these four expressions;

$$4a,\ 4a + 1,\ 4a + 2,\ 4a + 3.$$

62. It is nearly the same when the divisor is 5; for all numbers which can be divided by it are comprehended in the expression $5a$, and those which cannot be divided by 5, are reducible to one of the following expressions:

$$5a + 1,\ 5a + 2,\ 5a + 3,\ 5a + 4;$$

and we may go on in the same manner, and consider the greatest divisors.

63. It is proper to recollect here what has been already said on the resolution of numbers into their simple factors; for every number among the factors of which is found,

2, or 3, or 4, or 5, or 7,

or any other number, will be divisible by those numbers. For example; 60 being equal to $2 \times 2 \times 3 \times 5$, it is evident that 60 is divisible by 2, and by 3, and by 5.

64. Further, as the general expression $abcd$ is not only divisible by a, and b, and c, and d, but also by

ab, ac, ad, bc, bd, cd, and by
abc, abd, acd, bcd, and lastly by
$abcd$, that is to say, its own value;

it follows that 60, or $2 \times 2 \times 3 \times 5$, may be divided not only by these simple numbers, but also by those which are composed of two of them; that is to say, by 4, 6, 10, 15; and also by those which are composed of three of the simple factors, that is to say, by 12, 20, 30, and lastly by 60 itself.

65. *When, therefore, we have represented any number, assumed at pleasure, by its simple factors, it will be very easy to show all the numbers by which it is divisible. For we have only, first, to take the simple factors one by one, and then to multiply them together two by two, three by three, four by four, &c. till we arrive at the number proposed.*

66. It must here be particularly observed; that every number is divisible by 1; and also that every number is divisible by itself; so that every number has at least two factors, or divisors, the number

itself and unity; but every number, which has no other divisor than these two, belongs to the class of numbers, which we have before called *simple*, or *prime numbers*.

All numbers, except these, have, beside unity and themselves, other divisors, as may be seen from the following table, in which are placed under each number all its divisors.

TABLE.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	3	2	5	2	7	2	3	2	11	2	13	2	3	2	17	2	19	2
			4		3		4	9	5		3		7	5	4		3		4
					6		8		10		4		14	15	8		6		5
											6				16		9		10
											12						18		20
1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2	6
P.	P.	P.		P.		P.				P.		P.				P.		P.	

67. Lastly, it ought to be observed, that 0, or *nothing*, may be considered as a number which has the property of being divisible by all possible numbers; because by whatever number a we divide 0, the quotient is always 0; for it must be remarked that the multiplication of any number by *nothing* produces nothing, and therefore 0 times a, or 0 a, is 0.

CHAPTER VII.

Of Fractions in general.

68. When a number, as 7 for instance, is said not to be divisible by another number, let us suppose by 3, this only means, that the quotient cannot be expressed by an integral number; and it must not be thought by any means that it is impossible to form an idea of that quotient. Only imagine a line of 7 feet in length, no one can doubt the possibility of dividing this line into 3 equal parts, and of forming a notion of the length of one of those parts.

69. Since therefore we may form a precise idea of the quotient obtained in similar cases, though that quotient is not an integral number, this leads us to consider a particular species of numbers, called *fractions* or *broken numbers*. The instance adduced furnishes an illustration. If we have to divide 7 by 3, we easily conceive the quotient which should result, and express it by $\frac{7}{3}$; placing the divisor under the dividend, and separating the two numbers by a stroke or line.

70. So, *in general, when the number* a *is to be divided by the number* b, *we represent the quotient by* $\frac{a}{b}$ *and call this form of expression a fraction.* We cannot, therefore, give a better idea of a fraction $\frac{a}{b}$, than by saying that we thus express the quotient resulting from the division of the upper number by the lower. We must remember also, that in all fractions the lower number is called the *denominator*, and that above the line the *numerator*.

71. In the above fraction, $\frac{7}{3}$, which we read *seven thirds*, 7 is the numerator, and 3 the denominator. We must also read $\frac{2}{3}$, two thirds; $\frac{3}{4}$, three fourths; $\frac{3}{8}$, three eighths; $\frac{12}{100}$, twelve hundredths; and $\frac{1}{2}$, one half.

72. In order to obtain a more perfect knowledge of the nature of fractions, we shall begin by considering the case in which the numerator is equal to the denominator, as in $\frac{a}{a}$. Now, since this expresses the quotient obtained by dividing a by a, it is evident that this quotient is exactly unity, and that consequently this fraction $\frac{a}{a}$ is equal to 1, or one integer; for the same reason, all the following fractions,

$$\frac{2}{2},\ \frac{3}{3},\ \frac{4}{4},\ \frac{5}{5},\ \frac{6}{6},\ \frac{7}{7},\ \frac{8}{8},\ \&c.,$$

are equal to one another, each being equal to 1, or one integer.

73. We have seen that a fraction, whose numerator is equal to the denominator, is equal to unity. All fractions therefore, whose numerators are less than the denominators, have a value less than unity. For, if I have a number to be divided by another which is greater, the result must necessarily be less than 1; if we cut a line, for example, two feet long, into three parts, one of those parts will unquestionably be shorter than a foot; it is evident then, that $\frac{2}{3}$ is less than 1, for the same reason, that the numerator 2 is less than the denominator 3.

74. If the numerator, on the contrary, be greater than the denominator, the value of the fraction is greater than unity. Thus $\frac{3}{2}$ is greater than 1, for $\frac{3}{2}$ is equal to $\frac{2}{2}$ together with $\frac{1}{2}$. Now $\frac{2}{2}$ is exactly 1, consequently $\frac{3}{2}$ is equal to $1 + \frac{1}{2}$, that is, to an integer and a half. In the same manner $\frac{4}{3}$ is equal to $1\frac{1}{3}$, $\frac{5}{3}$ to $1\frac{2}{3}$, and $\frac{7}{3}$ to $2\frac{1}{3}$. And in general, it is sufficient in such cases to divide the upper number by the lower, and to add to the quotient a fraction having the remainder for the numerator, and the divisor for the denominator. If the given fraction were, for example, $\frac{43}{12}$, we should have for the quotient 3, and 7 for the remainder; whence we conclude that $\frac{43}{12}$ is the same as $3\frac{7}{12}$.

75. Thus we see how fractions, whose numerators are greater than the denominators, are resolved into two parts; one of which is an integer, and the other a fractional number, having the numerator less than the denominator. Such fractions as contain one or more integers, are called *improper fractions*, to distinguish them from fractions properly so called, which, having the numerator less than the denominator, are less than unity, or than an integer.

76. The nature of fractions is frequently considered in another way, which may throw additional light on the subject. If we consider, for example, the fraction $\frac{3}{4}$, it is evident that it is three times greater than $\frac{1}{4}$. Now this fraction $\frac{1}{4}$ means, that if we divide 1 into 4 equal parts, this will be the value of one of those parts; it is obvious then, that by taking 3 of those parts, we shall have the value of the fraction $\frac{3}{4}$.

In the same manner we may consider every other fraction; for example, $\frac{7}{12}$; if we divide unity into 12 equal parts, 7 of those parts will be equal to this fraction.

77. From this manner of considering fractions, the expressions *numerator* and *denominator* are derived. For, as in the preceding fraction $\frac{7}{12}$, the number under the line shows, that 12 is the number of parts into which unity is to be divided; and as it may be said to denote, or name the parts, it has not improperly been called the *denominator*.

Further, as the upper number, namely 7, shows that, in order to have the value of the fraction, we must take, or collect 7 of those parts, and therefore may be said to reckon, or number them, it has been thought proper to call the number above the line the *numerator*.

78. As it is easy to understand what $\frac{3}{4}$ is, when we know the signification of $\frac{1}{4}$, we may consider the fractions, whose numerator is

unity, as the foundation of all others. Such are the fractions,

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \&c.,$$

and it is observable that these fractions go on continually diminishing; for the more you divide an integer, or the greater the number of parts into which you distribute it, the less does each of those parts become. Thus $\frac{1}{100}$ is less than $\frac{1}{10}$; $\frac{1}{1000}$ is less than $\frac{1}{100}$; and $\frac{1}{10000}$ is less than $\frac{1}{1000}$.

79. As we have seen, that the more we increase the denominator of such fractions, the less their values become; it may be asked, whether it is not possible to make the denominator so great, that the fraction shall be reduced to nothing? I answer, no; for into whatever number of parts unity (the length of a foot for instance) is divided; let those parts be ever so small, they will still preserve a certain magnitude, and therefore can never be absolutely reduced to nothing.

80. It is true, if we divide the length of a foot into 1000 parts; those parts will not easily fall under the cognizance of our senses; but view them through a good microscope, and each of them will appear large enough to be subdivided into 100 parts and more.

At present, however, we have nothing to do with what depends on ourselves, or with what we are capable of performing, and what our eyes can perceive; the question is rather, what is possible in itself. And, in this sense of the word, it is certain, that however great we suppose the denominator, the fraction will never entirely vanish, or become equal to 0.

81. We never therefore arrive completely at nothing, however great the denominator may be; and these fractions always preserving a certain value, we may continue the series of fractions in the 78th article without interruption. This circumstance has introduced the expression, that the denominator must be *infinite*, or infinitely great, in order that the fraction may be reduced to 0, or to nothing; and the word *infinite* in reality signifies here, that we should never arrive at the end of the series of the above mentioned *fractions*.

82. To express this idea, which is extremely well founded, we make use of the sign ∞, which consequently indicates a number infinitely great; and we may therefore say that this fraction $\frac{1}{\infty}$ is really nothing, for the very reason that a fraction cannot be reduced to nothing, until the denominator has been increased to *infinity*.

83. It is the more necessary to pay attention to this idea of infinity, as it is derived from the first foundations of our knowledge, and as it will be of the greatest importance in the following part of this treatise.

We may here deduce from it a few consequences, that are extremely curious and worthy of attention. The fraction $\frac{1}{\infty}$ represents the quotient resulting from the division of the dividend 1 by the divisor ∞. Now we know that if we divide the dividend 1 by the quotient $\frac{1}{\infty}$, which is equal to 0, we obtain again the divisor ∞; hence we acquire a new idea of infinity; we learn that it arises from the division of 1 by 0; and we are therefore entitled to say, that 1 divided by 0 expresses a number infinitely great, or ∞.

84. It may be necessary also in this place to correct the mistake of those who assert, that a number infinitely great is not susceptible of increase. This opinion is inconsistent with the just principles which we have laid down; for $\frac{1}{0}$ signifying a number infinitely great, and $\frac{2}{0}$ being incontestably the double of $\frac{1}{0}$, it is evident that a number, though infinitely great, may still become two or more times greater.

CHAPTER VIII.

Of the Properties of Fractions.

85. We have already seen, that each of the fractions,

$$\frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \frac{7}{7}, \frac{8}{8}, \&c.$$

makes an integer, and that consequently they are all equal to one another. The same equality exists in the following fractions,

$$\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \frac{10}{5}, \frac{12}{6}, \&c.,$$

each of them making two integers; for the numerator of each, divided by its denominator, gives 2. So all the fractions

$$\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5}, \frac{18}{6}, \&c.,$$

are equal to one another, since 3 is their common value.

86. We may likewise represent the value of any fraction, in an infinite variety of ways. For *if we multiply both the numerator and the denominator of a fraction by the same number, which may be assumed at pleasure, this fraction will still preserve the same value.*

For this reason all the fractions

$$\tfrac{1}{2},\ \tfrac{2}{4},\ \tfrac{3}{6},\ \tfrac{4}{8},\ \tfrac{5}{10},\ \tfrac{6}{12},\ \tfrac{7}{14},\ \tfrac{8}{16},\ \tfrac{9}{18},\ \tfrac{10}{20},\ \&c.,$$

are equal, the value of each being $\frac{1}{2}$. Also

$$\tfrac{1}{3},\ \tfrac{2}{6},\ \tfrac{3}{9},\ \tfrac{4}{12},\ \tfrac{5}{15},\ \tfrac{6}{18},\ \tfrac{7}{21},\ \tfrac{8}{24},\ \tfrac{9}{27},\ \tfrac{10}{30},\ \&c.,$$

are equal fractions, the value of each of which is $\frac{1}{3}$. The fractions,

$$\tfrac{2}{3},\ \tfrac{4}{6},\ \tfrac{8}{12},\ \tfrac{10}{15},\ \tfrac{12}{18},\ \tfrac{14}{21},\ \tfrac{16}{24},\ \&c.,$$

have likewise all the same value; and lastly, we may conclude in general, that the fraction $\frac{a}{b}$ may be represented by the following expressions, each of which is equal to $\frac{a}{b}$; namely,

$$\frac{a}{b},\ \frac{2a}{2b},\ \frac{3a}{3b},\ \frac{4a}{4b},\ \frac{5a}{5b},\ \frac{6a}{6b},\ \frac{7a}{7b},\ \&c.$$

87. To be convinced of this we have only to write for the value of the fraction $\frac{a}{b}$ a certain letter c representing by this letter c the quotient of the division of a by b; and to recollect that the multiplication of the quotient c by the divisor b must give the dividend. For since c multiplied by b gives a, it is evident that c multiplied by $2\,b$ will give $2\,a$, that c multiplied by $3\,b$ will give $3\,a$, and that in general c multiplied by $m\,b$ must give $m\,a$. Now changing this into an example of division, and dividing the product $m\,a$, by $m\,b$ one of the factors, the quotient must be equal to the other factor c; but $m\,a$ divided by $m\,b$ gives also the fraction $\frac{m\,a}{m\,b}$, which is consequently equal to c; and this is what was to be proved: for c having been assumed as the value of the fraction $\frac{a}{b}$, it is evident that this fraction is equal to the fraction $\frac{m\,a}{m\,b}$, whatever be the value of m.

88. We have seen that *every fraction may be represented in an infinite number of forms*, each of which contains the same value; and it is evident that of all these forms, that, which shall be composed of the least numbers, will be most easily understood. For example, we might substitute instead of $\frac{2}{3}$ the following fractions,

$$\tfrac{4}{6},\ \tfrac{6}{9},\ \tfrac{8}{12},\ \tfrac{10}{15},\ \tfrac{12}{18},\ \&c.;$$

but of all these expressions $\frac{2}{3}$ is that of which it is easiest to form an idea. Here, therefore, a problem arises; how a fraction, such as

$\frac{6}{15}$, which is not expressed by the least possible numbers, may be reduced to its simplest form, or to *its least terms*, that is to say, in our present example, to $\frac{2}{5}$.

89. It will be easy to resolve this problem, if we consider that a fraction still preserves its value, when we multiply both its terms, or its numerator and denominator, by the same number. For from this it follows also, that *if we divide the numerator and denominator of a fraction by the same number, the fraction still preserves the same value*. This is made more evident by means of the general expression $\frac{ma}{mb}$; for if we divide both the numerator $m\,a$ and the denominator $m\,b$ by the number m, we obtain the fraction $\frac{a}{b}$, which, as was before proved, is equal to $\frac{ma}{mb}$.

90. In order, therefore, to reduce a given fraction to its least terms, it is required to find a number by which both the numerator and denominator may be divided. Such a number is called a *common divisor*, and so long as we can find a common divisor to the numerator and the denominator, it is certain that the fraction may be reduced to a lower form; but, on the contrary, when we see that except unity no other common divisor can be found, this shows that the fraction is already in the simplest form that it admits of.

91. To make this more clear, let us consider the fraction $\frac{48}{120}$. We see immediately that both the terms are divisible by 2, and that there results the fraction $\frac{24}{60}$. Then that it may again be divided by 2, and reduced to $\frac{12}{30}$; and this also, having 2 for a common divisor, it is evident, may be reduced to $\frac{6}{15}$. But now we easily perceive, that the numerator and denominator are still divisible by 3; performing this division, therefore, we obtain the fraction $\frac{2}{5}$, which is equal to the fraction proposed, and gives the simplest expression to which it can be reduced; for 2 and 5 have no common divisor but 1, which cannot diminish these numbers any further.

92. This property of fractions preserving an invariable value, whether we divide or multiply the numerator and denominator by the same number, is of the greatest importance, and is the principal foundation of the doctrine of fractions. For example, we can scarcely add together two fractions, or subtract them from each other, before we have, by means of this property, reduced them to

other forms, that is to say, to expressions whose denominators are equal. Of this we shall treat in the following chapter.

93. We conclude the present by remarking, that all integers may also be represented by fractions. For example, 6 is the same as $\frac{6}{1}$, because 6 divided by 1 makes 6; and we may, in the same manner, express the number 6 by the fractions $\frac{12}{2}$, $\frac{18}{3}$, $\frac{24}{4}$, $\frac{36}{6}$, and an infinite number of others, which have the same value.

CHAPTER IX.

Of the Addition and Subtraction of Fractions.

94. When fractions have equal denominators, there is no difficulty in adding and subtracting them; for $\frac{2}{7} + \frac{3}{7}$ is equal to $\frac{5}{7}$, and $\frac{4}{7} - \frac{2}{7}$ is equal to $\frac{2}{7}$. In this case, either for addition or subtraction, we alter only the numerators, and place the common denominator under the line; thus,

$\frac{7}{100} + \frac{9}{100} - \frac{12}{100} - \frac{15}{100} + \frac{20}{100}$ is equal to $\frac{9}{100}$; $\frac{24}{50} - \frac{7}{50} - \frac{12}{50} + \frac{31}{50}$ is equal to $\frac{36}{50}$, or $\frac{18}{25}$; $\frac{16}{20} - \frac{3}{20} - \frac{11}{20} + \frac{14}{20}$ is equal to $\frac{16}{20}$, or $\frac{4}{5}$; also $\frac{1}{3} + \frac{2}{3}$ is equal to $\frac{3}{3}$, or 1, that is to say, an integer; and $\frac{2}{4} - \frac{3}{4} + \frac{1}{4}$ is equal to $\frac{0}{4}$, that is to say, nothing, or 0.

95. But *when fractions have not equal denominators, we can always change them into other fractions that have the same denominator.* For example, when it is proposed to add together the fractions $\frac{1}{2}$ and $\frac{1}{3}$, we must consider that $\frac{1}{2}$ is the same as $\frac{3}{6}$, and that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$; we have therefore, instead of the two fractions proposed, these $\frac{3}{6} + \frac{2}{6}$, the sum of which is $\frac{5}{6}$. If the two fractions were united by the sign *minus*, as $\frac{1}{2} - \frac{1}{3}$, we should have $\frac{3}{6} - \frac{2}{6}$ or $\frac{1}{6}$.

Another example: let the fractions proposed be $\frac{3}{4} + \frac{5}{8}$; since $\frac{3}{4}$ is the same as $\frac{6}{8}$, this value may be substituted for it, and we may say $\frac{6}{8} + \frac{5}{8}$ makes $\frac{11}{8}$, or $1\frac{3}{8}$.

Suppose further, that the sum of $\frac{1}{3}$ and $\frac{1}{4}$ were required. I say that it is $\frac{7}{12}$; for $\frac{1}{3}$ makes $\frac{4}{12}$, and $\frac{1}{4}$ makes $\frac{3}{12}$.

96. *We may have a greater number of fractions to be reduced to a common denominator;* for example, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$; *in this case the whole depends on finding a number which may be divisible by*

all the denominators of these fractions. In this instance 60 is the number which has that property, and which consequently becomes the common denominator. We shall therefore have $\frac{30}{60}$ instead of $\frac{1}{2}$; $\frac{40}{60}$ instead of $\frac{2}{3}$; $\frac{45}{60}$ instead of $\frac{3}{4}$; $\frac{48}{60}$ instead of $\frac{4}{5}$; and $\frac{50}{60}$ instead of $\frac{5}{6}$. *If now it be required to add together all these fractions* $\frac{30}{60}$, $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$, and $\frac{50}{60}$, *we have only to add all the numerators, and under the sum place the common denominator* 60; that is to say, we shall have $\frac{213}{60}$, or three integers, and $\frac{33}{60}$, or $3\frac{11}{20}$.

97. The whole of this operation consists, as we before stated, in changing two fractions, whose denominators are unequal, into two others, whose denominators are equal. In order therefore to perform it generally, let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions proposed. First, multiply the two terms of the first fraction by d, we shall have the fraction $\frac{ad}{bd}$ equal to $\frac{a}{b}$; next multiply the two terms of the second fraction by b, and we shall have an equivalent value of it expressed by $\frac{bc}{bd}$; thus the two denominators become equal. Now if the sum of the two proposed fractions be required, we may immediately answer that it is $\frac{ad+bc}{bd}$; and if their difference be asked, we say that it is $\frac{ad-bc}{bd}$. If the fractions $\frac{5}{8}$ and $\frac{7}{9}$, for example, were proposed, we should obtain in their stead $\frac{45}{72}$ and $\frac{56}{72}$; of which the sum is $\frac{101}{72}$, and the difference $\frac{11}{72}$.

98. To this part of the subject belongs also the question, of two proposed fractions, which is the greater or the less; for, to resolve this, we have only to reduce the two fractions to the same denominator. Let us take, for example, the two fractions $\frac{2}{3}$ and $\frac{5}{7}$: when reduced to the same denominator, the first becomes $\frac{14}{21}$, and the second $\frac{15}{21}$, and it is evident that the second, or $\frac{5}{7}$, is the greater, and exceeds the former by $\frac{1}{21}$.

Again, let the two fractions $\frac{3}{5}$ and $\frac{5}{8}$ be proposed. We shall have to substitute for them $\frac{24}{40}$ and $\frac{25}{40}$; whence we may conclude that $\frac{5}{8}$ exceeds $\frac{3}{5}$, but only by $\frac{1}{40}$.

99. *When it is required to subtract a fraction from an integer*, it is sufficient to *change one of the units of that integer into a fraction having the same denominator as the fraction to be subtracted;* in the rest of the operation there is no difficulty. If it be required,

for example, to subtract $\frac{2}{3}$ from 1, we write $\frac{3}{3}$ instead of 1, and say, that $\frac{2}{3}$ taken from $\frac{3}{3}$ leaves the remainder $\frac{1}{3}$. So $\frac{5}{12}$ subtracted from 1, leaves $\frac{7}{12}$.

If it were required to subtract $\frac{3}{4}$ from 2, we should write 1 and $\frac{4}{4}$ instead of 2, and we should immediately see that after the subtraction there must remain $1\frac{1}{4}$.

100. It happens also sometimes, that having added two or more fractions together, we obtain more than an integer; that is to say, a numerator greater than the denominator: this is a case which has already occurred, and deserves attention.

We found, for example, article 96, that the sum of the five fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$, was $\frac{213}{60}$, and we remarked that the value of this sum was 3 integers and $\frac{33}{60}$, or $\frac{11}{20}$. Likewise $\frac{2}{3}+\frac{3}{4}$, or $\frac{8}{12}+\frac{9}{12}$, makes $\frac{17}{12}$, or $1\frac{5}{12}$. We have only to perform the actual division of the numerator by the denominator, to see how many integers there are for the quotient, and to set down the remainder. Nearly the same must be done to add together numbers compounded of integers and fractions; we first add the fractions, and if their sum produces one or more integers, these are added to the other integers. Let it be proposed, for example, to add $3\frac{1}{2}$ and $2\frac{2}{3}$; we first take the sum of $\frac{1}{2}$ and $\frac{2}{3}$, or of $\frac{3}{6}$ and $\frac{4}{6}$. It is $\frac{7}{6}$ or $1\frac{1}{6}$; then the sum total is $6\frac{1}{6}$.

CHAPTER X.

Of the Multiplication and Division of Fractions.

101. The *rule for the multiplication of a fraction by an integer, or whole number, is to multiply the numerator only by the given number, and not to change the denominator:* thus,

2 times, or twice $\frac{1}{2}$ makes $\frac{2}{2}$, or 1 integer;
2 times, or twice $\frac{1}{3}$ makes $\frac{2}{3}$;
3 times, or thrice $\frac{1}{6}$ makes $\frac{3}{6}$, or $\frac{1}{2}$; and
4 times $\frac{5}{12}$ makes $\frac{20}{12}$ or $1\frac{8}{12}$, or $1\frac{2}{3}$.

But, instead of this rule, we may use that of dividing the denominator by the given integer; and this is preferable, when it can be *used, because it shortens the operation.* Let it be required, for example, to multiply $\frac{8}{9}$ by 3; if we multiply the numerator by the given integer we obtain $\frac{24}{9}$, which product we must reduce to $\frac{8}{3}$. But if we do not change the numerator, and divide the denominator by the integer, we find immediately $\frac{8}{3}$, or $2\frac{2}{3}$ for the given product. Likewise $\frac{13}{24}$ multiplied by 6 gives $\frac{13}{4}$, or $3\frac{1}{4}$.

102. In general, therefore, the product of the multiplication of a fraction $\frac{a}{b}$ by c is $\frac{ac}{b}$; and it may be remarked, *when the integer is exactly equal to the denominator*, that *the product must be equal to the numerator.*

So that $\begin{cases} \frac{1}{2} \text{ taken twice gives } 1; \\ \frac{2}{3} \text{ taken thrice gives } 2; \\ \frac{3}{4} \text{ taken 4 times gives } 3. \end{cases}$

And in general, if we multiply the fraction $\frac{a}{b}$ by the number b, the product must be a, as we have already shown; for since $\frac{a}{b}$ expresses the quotient resulting from the division of the dividend a by the divisor b, and since it has been demonstrated that the quotient multiplied by the divisor will give the dividend, it is evident that $\frac{a}{b}$ multiplied by b must produce a.

103. We have shown how a fraction is to be multiplied by an integer; let us now consider also *how a fraction is to be divided by an integer;* this inquiry is necessary before we proceed to the multiplication of fractions by fractions. It is evident, if I have to divide the fraction $\frac{2}{3}$ by 2, that the result must be $\frac{1}{3}$; and that the quotient of $\frac{6}{7}$ divided by 3 is $\frac{2}{7}$. The rule therefore is, to *divide the numerator by the integer without changing the denominator.* Thus,

$\frac{12}{25}$ divided by 2 gives $\frac{6}{25}$;
$\frac{12}{25}$ divided by 3 gives $\frac{4}{25}$; and
$\frac{12}{25}$ divided by 4 gives $\frac{3}{25}$; &c.

104. This rule may be easily practised, provided the numerator be divisible by the number proposed; but very often it is not: it must therefore be observed that a fraction may be transformed into an infinite number of other expressions, and in that number there must be some by which the numerator might be divided by the given integer. If it were required, for example, to divide $\frac{3}{4}$ by 2,

we should change the fraction into $\frac{6}{8}$, and then dividing the numerator by 2, we should immediately have $\frac{3}{8}$ for the quotient sought.

In general, if it be proposed to divide the fraction $\frac{a}{b}$ by c, we change it into $\frac{ac}{bc}$, and then dividing the numerator ac by c, write $\frac{a}{bc}$ for the quotient sought.

105. *When therefore a fraction* $\frac{a}{b}$ *is to be divided by an integer* c, *we have only to multiply the denominator by that number, and leave the numerator as it is.* Thus $\frac{5}{8}$ divided by 3 gives $\frac{5}{24}$, and $\frac{9}{16}$ divided by 5 gives $\frac{9}{80}$.

This operation becomes easier when the numerator itself is divisible by the integer, as we have supposed in article 103. For example, $\frac{9}{16}$ divided by 3 would give, according to our last rule, $\frac{9}{48}$; but by the first rule, which is applicable here, we obtain $\frac{3}{16}$, an expression equivalent to $\frac{9}{48}$, but more simple.

106. We shall now be able to understand how one fraction $\frac{a}{b}$ may be multiplied by another fraction $\frac{c}{d}$. We have only to consider that $\frac{c}{d}$ means that c is divided by d; and on this principle, we shall first multiply the fraction $\frac{a}{b}$ by c, which produces the result $\frac{ac}{b}$; after which we shall divide by d, which gives $\frac{ac}{bd}$.

Hence the following rule for multiplying fractions; multiply separately the numerators and the denominators.

Thus $\frac{1}{2}$ by $\frac{2}{3}$ gives the product $\frac{2}{6}$, or $\frac{1}{3}$;
$\frac{2}{3}$ by $\frac{4}{5}$ makes $\frac{8}{15}$;
$\frac{3}{4}$ by $\frac{5}{12}$ produces $\frac{15}{48}$, or $\frac{5}{16}$; &c.

107. It remains to show *how one fraction may be divided by another.* We remark first, that *if the two fractions have the same number for a denominator, the division takes place only with respect to the numerators*; for it is evident, that $\frac{3}{12}$ is contained as many times in $\frac{9}{12}$ as 3 in 9, that is to say, thrice; and in the same manner, in order to divide $\frac{8}{12}$ by $\frac{9}{12}$, we have only to divide 8 by 9, which gives $\frac{8}{9}$. We shall also have $\frac{6}{20}$ in $\frac{18}{20}$, 3 times: $\frac{7}{100}$ in $\frac{49}{100}$, 7 times; $\frac{7}{25}$ in $\frac{6}{25}$, $\frac{6}{7}$; &c.

108. But *when the fractions have not equal denominators,* we must have recourse to the method already mentioned for reducing them to a common denominator. Let there be, for example, the fraction $\frac{a}{b}$ to be divided by the fraction $\frac{c}{d}$; we first reduce them to the same denominator; we have then $\frac{ad}{bd}$ to be divided by $\frac{bc}{bd}$; it is now evident that the quotient must be represented simply by the division of ad by bc; which gives $\frac{ad}{bc}$.

Hence the following rule: ***Multiply the numerator of the dividend by the denominator of the divisor, and the denominator of the dividend by the numerator of the divisor; the first product will be the numerator of the quotient, and the second will be its denominator.***

109. Applying this rule to the division of $\frac{3}{8}$ by $\frac{2}{3}$, we shall have the quotient $\frac{9}{16}$; the division of $\frac{3}{4}$ by $\frac{1}{2}$ will give $\frac{6}{4}$ or $\frac{3}{2}$ or 1 and $\frac{1}{2}$; and $\frac{25}{48}$ by $\frac{5}{6}$ will give $\frac{150}{240}$, or $\frac{5}{8}$.

110. This rule for division is often represented in a manner more easily remembered, as follows: *Invert the fraction which is the divisor, so that the denominator may be in the place of the numerator, and the latter be written under the line; then multiply the fraction, which is the dividend by this inverted fraction, and the product will be the quotient sought.* Thus $\frac{3}{4}$ divided by $\frac{1}{2}$ is the same as $\frac{3}{4}$ multiplied by $\frac{2}{1}$, which makes $\frac{6}{4}$, or $1\frac{1}{2}$. Also $\frac{3}{8}$ divided by $\frac{2}{3}$ is the same as $\frac{3}{8}$ multiplied by $\frac{3}{2}$, which is $\frac{9}{16}$; or $\frac{25}{48}$ divided by $\frac{5}{6}$ gives the same as $\frac{25}{48}$ multiplied by $\frac{6}{5}$, the product of which is $\frac{150}{240}$, or $\frac{5}{8}$.

We see then, in general, that *to divide by the fraction $\frac{1}{2}$, is the same as to multiply by $\frac{2}{1}$, or 2; that division by $\frac{1}{3}$ amounts to multiplication by $\frac{3}{1}$, or by 3, &c.*

111. The number 100 divided by $\frac{1}{2}$ will give 200; and 1000 divided by $\frac{1}{3}$ will give 3000. Further, if it were required to divide 1 by $\frac{1}{1000}$, the quotient would be 1000; and dividing 1 by $\frac{1}{100000}$, the quotient is 100000. This enables us to conceive that, when any number is divided by 0, the result must be a number infinitely great; for even the division of 1 by the small fraction $\frac{1}{1000000000}$ gives for the quotient the very great number 1000000000.

112. Every number when divided by itself producing unity, it is evident that a fraction divided by itself must give 1 for the quotient. The same follows from our rule: for, in order to divide $\frac{3}{4}$ by $\frac{3}{4}$, we must multiply $\frac{3}{4}$ by $\frac{4}{3}$, and we obtain $\frac{12}{12}$, or 1; and if it

be required to divide $\frac{a}{b}$ by $\frac{a}{b}$, we multiply $\frac{a}{b}$ by $\frac{b}{a}$; now the product $\frac{ab}{ab}$ is equal to 1.

113. We have still to explain an expression which is frequently used. It may be asked, for example, what is the half of $\frac{3}{4}$; this means that we must multiply $\frac{3}{4}$ by $\frac{1}{2}$. So likewise, if the value of $\frac{2}{3}$ of $\frac{5}{8}$ were required, we should multiply $\frac{5}{8}$ by $\frac{2}{3}$, which produces $\frac{10}{24}$; and $\frac{3}{4}$ of $\frac{9}{16}$ is the same as $\frac{9}{16}$ multiplied by $\frac{3}{4}$, which produces $\frac{27}{64}$.

114. Lastly, we must here observe the same rules with respect to the signs $+$ and $-$, that we before laid down for integers. Thus $+\frac{1}{2}$ multiplied by $-\frac{1}{3}$ makes $-\frac{1}{6}$; and $-\frac{2}{3}$ multiplied by $-\frac{4}{5}$ gives $+\frac{8}{15}$. Farther, $-\frac{5}{8}$ divided by $+\frac{2}{3}$ makes $-\frac{15}{16}$; and $-\frac{3}{4}$ divided by $-\frac{3}{4}$ makes $+\frac{12}{12}$ or $+1$.

CHAPTER XI.

Of Square Numbers.

115. The *product of a number, when multiplied by itself, is called a square; and for this reason, the number, considered in relation to such a product, is called a square root.*

For example, when we multiply 12 by 12, the product 144 is a square, of which the root is 12.

This term is derived from geometry, which teaches us that the contents of a square are found by multiplying its side by itself.

116. Square numbers are found therefore by multiplication; that is to say, by multiplying the root by itself. Thus 1 is the square of 1, since 1 multiplied by 1 makes 1; likewise, 4 is the square of 2; and 9 the square of 3; 2 also is the root of 4, and 3 is the root of 9.

We shall begin by considering the squares of natural numbers, and shall first give the following small table, on the first line of which several numbers, or roots are placed, and on the second their squares.

Numbers	1	2	3	4	5	6	7	8	9	10	11	12	13
Squares	1	4	9	16	25	36	49	64	81	100	121	144	169

117. It will be readily perceived, that the series of square numbers thus arranged has a singular property; namely, that if each of them be subtracted from that which immediately follows, the remainders always increase by 2, and form this series:

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, &c.

118. *The squares of fractions are found in the same manner, by multiplying any given fraction by itself.* For example, the square of $\frac{1}{2}$ is $\frac{1}{4}$,

$$\text{The square of } \begin{cases} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{4} \\ \frac{3}{4} \end{cases} \text{ is } \begin{cases} \frac{1}{9}; \\ \frac{4}{9}; \\ \frac{1}{16}; \\ \frac{9}{16}; \text{ \&c.} \end{cases}$$

We have only, therefore, to divide the square of the numerator by the square of the denominator, and the fraction, which expresses that division must be the square of the given fraction. Thus, $\frac{25}{64}$ is the square of $\frac{5}{8}$; and reciprocally, $\frac{5}{8}$ is the root of $\frac{25}{64}$.

119. When the square of a mixed number, or a number composed of an integer and a fraction, is required, we have only to reduce it to a single fraction, and then to take the square of that fraction. Let it be required, for example, to find the square of $2\frac{1}{2}$; we first express this number by $\frac{5}{2}$, and taking the square of that fraction, we have $\frac{25}{4}$, or $6\frac{1}{4}$, for the value of the square of $2\frac{1}{2}$. So to obtain the square of $3\frac{1}{4}$, we say $3\frac{1}{4}$ is equal to $\frac{13}{4}$; therefore its square is equal to $\frac{169}{16}$, or to 10 and $\frac{9}{16}$. The squares of the numbers between 3 and 4, supposing them to increase by one fourth, are as follows:

Numbers	3	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	4
Squares	9	$10\frac{9}{16}$	$12\frac{1}{4}$	$14\frac{1}{16}$	16

From this small table we may infer, that if a root contain a fraction, its square also contains one. Let the root, for example, be $1\frac{5}{12}$; its square is $\frac{289}{144}$, or $2\frac{1}{144}$; that is to say, a little greater than the integer 2.

120. Let us proceed to general expressions. When the root is a, the square must be $a\,a$; if the root be $2\,a$, the square is $4\,a\,a$; which shows that by doubling the root, the square becomes 4 times greater. So if the root be $3\,a$, the square is $9\,a\,a$; and if the root

be 4 a, the square is 16 $a\,a$. But if the root be $a\,b$, the square is $a\,a\,b\,b$; and if the root be $a\,b\,c$, the square is $a\,a\,b\,b\,c\,c$.

121. Thus *when the root is composed of two or more factors, we multiply their squares together; and reciprocally, if a square be composed of two or more factors, of which each is a square, we have only to multiply together the roots of those squares, to obtain the complete root of the square proposed.* Thus, as 2304 is equal to $4 \times 16 \times 36$, the square root of it is $2 \times 4 \times 6$, or 48; and 48 is found to be the true square root of 2304, because 48×48 gives 2304.

122. Let us now consider what rule is to be observed with regard to the signs + and —. First, it is evident that if the root has the sign +, that is to say, is a positive number, its square must necessarily be a positive number also, because + by + makes +: the square of $+ a$ will be $+ a\,a$. But if the root be a negative number, as $- a$, the square is still positive, for it is $+ a\,a$; we may therefore conclude, that $+ a\,a$ *is the square both of* $+$ a, *and of* $-$ a, and that *consequently every square has two roots, one positive and the other negative.* The square root of 25, for example, is both + 5 and — 5, because — 5 multiplied by — 5 gives 25, as well as + 5 by + 5.

CHAPTER XII.

Of Square Roots, and of Irrational Numbers resulting from them.

124. What we have said in the preceding chapter is chiefly this: that the square root of a given number is nothing but a number whose square is equal to the given number; and that we may put before these roots either the positive or the negative sign.

124. So that when a square number is given, provided we retain in our memory a sufficient number of square numbers, it is easy to find its root. If 196, for example, be the given number, we know that its square root is 14.

Fractions likewise are easily managed; it is evident, for example, that $\frac{5}{7}$ is the square root of $\frac{25}{49}$. To be convinced of this we have only to take the square root of the numerator, and that of the denominator.

If the number proposed be a mixed number, as $12\frac{1}{4}$, we reduce it to a single fraction, which here is $\frac{49}{4}$; and we immediately perceive that $\frac{7}{2}$, or $3\frac{1}{2}$, must be the square root of $12\frac{1}{4}$.

125. But when the given number is not a square, as 12, for example, it is not possible to extract its square root; or to find a number, which, multiplied by itself, will give the product 12. We know, however, that the square root of 12 must be greater than 3, because 3×3 produces only 9: and less than 4, because 4×4 produces 16, which is more than 12. We know also, that this root is less than $3\frac{1}{2}$; for we have seen that the square of $3\frac{1}{2}$, or $\frac{7}{2}$ is $12\frac{1}{4}$. Lastly, we may approach still nearer to this root, by comparing it with $3\frac{7}{15}$; for the square of $3\frac{7}{15}$, or of $\frac{52}{15}$ is $\frac{2704}{225}$, or $12\frac{4}{225}$, so that this fraction is still greater than the root required; but very little greater, as the difference of the two squares is only $\frac{4}{225}$.

126. We may suppose that as $3\frac{1}{2}$ and $3\frac{7}{15}$ are numbers greater than the root of 12, it might be possible to add to 3 a fraction a little less than $\frac{7}{15}$, and precisely such that the square of the sum would be equal to 12.

Let us therefore try with $3\frac{3}{7}$, since $\frac{3}{7}$ is a little less than $\frac{7}{15}$. Now $3\frac{3}{7}$ is equal to $\frac{24}{7}$, the square of which is $\frac{576}{49}$, and consequently less by $\frac{12}{49}$ than 12, which may be expressed by $\frac{588}{49}$. It is therefore proved that $3\frac{3}{7}$ is less, and that $3\frac{7}{15}$ is greater than the root required. Let us then try a number a little greater than $3\frac{3}{7}$, but yet less than $3\frac{7}{15}$, for example, $3\frac{5}{11}$. This number, which is equal to $\frac{38}{11}$, has for its square $\frac{1444}{121}$. Now, by reducing 12 to this denominator, we obtain $\frac{1452}{121}$; which shows that $3\frac{5}{11}$ is still less than the root of 12, *viz.* by $\frac{8}{121}$. Let us therefore substitute for $\frac{5}{11}$ the fraction $\frac{6}{13}$. which is a little greater, and see what will be the result of the comparison of the square of $3\frac{6}{13}$ with the proposed number 12. The square of $3\frac{6}{13}$ is $\frac{2025}{169}$; now 12 reduced to the same denominator is $\frac{2028}{169}$; so that $3\frac{6}{13}$ is still too small, though only by $\frac{3}{169}$, whilst $3\frac{7}{15}$ has been found too great.

127. It is evident, therefore, that whatever fraction be joined to 3, the square of that sum must always contain a fraction, and can never be exactly equal to the integer 12. Thus, although we know that the square root of 12 is greater than $3\frac{6}{13}$ and less than $3\frac{7}{15}$, yet we are unable to assign an intermediate fraction between these two, which, at the same time, if added to 3, would express exactly

the square root of 12. Notwithstanding this, we are not to assert that the square root of 12 is absolutely and in itself indeterminate; it only follows from what has been said, that this root, though it necessarily has a determinate magnitude, cannot be expressed by fractions.

128. *There is, therefore, a sort of numbers which cannot be assigned by fractions, and which are nevertheless determinate quantities*; the square root of 12 furnishes an example. We call this new species of numbers, *irrational numbers*; they occur whenever we endeavor to find the square root of a number which is not a square. Thus, 2 not being a perfect square, the square root of 2, or the number which, multiplied by itself, would produce 2, is an irrational quantity. These numbers are also called *surd quantities*, or *incommensurables*.

129. These irrational quantities, though they cannot be expressed by fractions, are nevertheless magnitudes, of which we may form an accurate idea. For however concealed the square root of 12, for example, may appear, we are not ignorant, that it must be a number which, when multiplied by itself, would exactly produce 12; and this property is sufficient to give us an idea of the number, since it is in our power to approximate its value continually.

130. As we are therefore sufficiently acquainted with the nature of the irrational numbers, under our present consideration, a particular sign has been agreed on, to express the square roots of all numbers that are not perfect squares. This sign is written thus $\sqrt{}$, and is read *square root*. Thus, $\sqrt{12}$ represents the square root of 12, or the number which, multiplied by itself, produces 12. So, $\sqrt{2}$ represents the square root of 2; $\sqrt{3}$ that of 3; $\sqrt{\frac{2}{3}}$ that of $\frac{2}{3}$, and in general, $\sqrt{a}$ *represents the square root of the number* a. Whenever, therefore, we would express the square root of a number which is not a square, we need only make use of the mark $\sqrt{}$ by placing it before the number.

131. The explanation which we have given of irrational numbers will readily enable us to apply to them the known methods of calculation. For knowing that the square root of 2, multiplied by itself, must produce 2; we know also, that the multiplication $\sqrt{2}$ by $\sqrt{2}$ must necessarily produce 2; that, in the same manner, the multiplication of $\sqrt{3}$ by $\sqrt{3}$ must give 3; that $\sqrt{5}$ by $\sqrt{5}$ makes 5; that $\sqrt{\frac{2}{3}}$ by $\sqrt{\frac{2}{3}}$ makes $\frac{2}{3}$; and, in general, that $\sqrt{a}$ *multiplied by* $\sqrt{a}$ *produces* a.

132. *But when it is required to multiply* $\sqrt{a}$ *by* $\sqrt{b}$ *the product will be found to be* $\sqrt{ab}$; because we have shown before, that if a square has two or more factors, its root must be composed of the roots of those factors. Wherefore we find the square root of the product $a\,b$, which is $\sqrt{ab}$, by multiplying the square root of a or $\sqrt{a}$, by the square root of b or $\sqrt{b}$. It is evident from this, that if b were equal to a, we should have $\sqrt{aa}$ for the product of $\sqrt{a}$ by $\sqrt{b}$. Now $\sqrt{aa}$ is evidently a, since $a\,a$ is the square of a.

133. In division, if it were required to divide $\sqrt{a}$, for example, by $\sqrt{b}$, we obtain $\sqrt{\frac{a}{b}}$; and in this instance the irrationality may vanish in the quotient. Thus, having to divide $\sqrt{18}$ by $\sqrt{8}$, the quotient is $\sqrt{\frac{18}{8}}$, which is reduced to $\sqrt{\frac{9}{4}}$, and consequently to $\frac{3}{2}$, because $\frac{9}{4}$ is the square of $\frac{3}{2}$

134. When the number, before which we have placed the radical sign $\sqrt{\ }$, is itself a square, its root is expressed in the usual way. Thus $\sqrt{4}$ is the same as 2; $\sqrt{9}$ the same as 3; $\sqrt{36}$ the same as 6; and $\sqrt{12\frac{1}{4}}$ the same as $\frac{7}{2}$, or $3\frac{1}{2}$. In these instances the irrationality is only apparent, and vanishes of course.

135. It is easy also to multiply irrational numbers by ordinary numbers. For example, 2 *multiplied by* $\sqrt{5}$ *makes* $2\sqrt{5}$, and 3 times $\sqrt{2}$ make $3\sqrt{2}$. In the second example, however, as 3 is equal to $\sqrt{9}$, we may also express 3 times $\sqrt{2}$ by $\sqrt{9}$ times $\sqrt{2}$, or by $\sqrt{18}$. So $2\sqrt{a}$ is the same as $\sqrt{4a}$, and $3\sqrt{a}$ the same as $\sqrt{9a}$. And, in general, $b\sqrt{a}$ *has the same value as the square root of* b b a, *or* $\sqrt{aab}$; whence we infer reciprocally, that when the number which is preceded by the radical sign contains a square, we may take the root of that square and put it before the sign, as we should do in writing $b\sqrt{a}$ instead of $\sqrt{abb}$. After this, the following reductions will be easily understood:

$$\left.\begin{array}{l}\sqrt{8},\text{ or }\sqrt{2.4}\\ \sqrt{12},\text{ or }\sqrt{3.4}\\ \sqrt{18},\text{ or }\sqrt{2.9}\\ \sqrt{24},\text{ or }\sqrt{6.4}\\ \sqrt{32},\text{ or }\sqrt{2.16}\\ \sqrt{75},\text{ or }\sqrt{3.25}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}2\sqrt{2};\\ 2\sqrt{3};\\ 3\sqrt{2};\\ 2\sqrt{6};\\ 4\sqrt{2};\\ 5\sqrt{3};\end{array}\right.$$

and so on.

136. Division is founded on the same principles. $\sqrt{a}$ *divided by* $\sqrt{b}$, *gives* $\frac{\sqrt{a}}{\sqrt{b}}$, *or* $\sqrt{\frac{a}{b}}$. In the same manner,

$$\left.\begin{array}{c}\frac{\sqrt{8}}{\sqrt{2}}\\ \frac{\sqrt{18}}{\sqrt{2}}\\ \frac{\sqrt{12}}{\sqrt{3}}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}\sqrt{\frac{8}{2}},\text{ or }\sqrt{4}\text{ or }2;\\ \sqrt{\frac{18}{2}},\text{ or }\sqrt{9},\text{ or }3;\\ \sqrt{\frac{12}{3}},\text{ or }\sqrt{4},\text{ or }2.\end{array}\right.$$

Further
$$\left.\begin{array}{c}\frac{2}{\sqrt{2}}\\ \frac{3}{\sqrt{3}}\\ \frac{12}{\sqrt{6}}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}\frac{\sqrt{4}}{\sqrt{2}},\text{ or }\sqrt{\tfrac{4}{2}},\text{ or }\sqrt{2};\\ \frac{\sqrt{9}}{\sqrt{3}},\text{ or }\sqrt{\tfrac{9}{3}},\text{ or }\sqrt{3};\\ \frac{\sqrt{144}}{\sqrt{6}};\text{ or }\sqrt{\tfrac{144}{6}},\text{ or }\sqrt{24},\end{array}\right.$$

or $\sqrt{6.4}$, or lastly $2\sqrt{6}$.

137. There is nothing in particular to be observed with respect to the addition and subtraction of such quantities, because we only connect them by the signs + and —. For example, $\sqrt{2}$ *added to* $\sqrt{3}$ *is written* $\sqrt{2}+\sqrt{3}$; *and* $\sqrt{3}$ *subtracted from* $\sqrt{5}$ *is written* $\sqrt{5}-\sqrt{3}$.

138. We may observe lastly, that in order to distinguish irrational numbers, we call all other numbers, both integral and fractional, *rational numbers*.

So that, whenever we speak of rational numbers, we understand integers or fractions.

CHAPTER XIII.

Of Impossible or Imaginary Quantities, which arise from the same source.

139. We have already seen that the squares of numbers, negative as well as positive, are always positive, or affected with the sign +; having shown that $-a$ multiplied by $-a$ gives $+aa$, the same as the product of $+a$ by $+a$. Wherefore, in the preceding chapter, we supposed that all the numbers, of which it was required to extract the square roots, were positive.

140. When it is required, therefore, to extract the root of a negative number, a very great difficulty arises; since there is no assign-

able number, the square of which would be a negative quantity. Suppose, for example, that we wished to extract the root of — 4; we require such a number, as when multiplied by itself, would produce — 4; now this number is neither + 2 nor — 2, because the square, both of + 2 and of — 2, is + 4, and not — 4.

141. We must therefore conclude, that *the square root of a negative number cannot be either a positive number, or a negative number*, since the squares of negative numbers also take the sign *plus*. Consequently the root in question must belong to an entirely distinct species of numbers; since it cannot be ranked either among positive or among negative numbers.

142. Now, we before remarked, that positive numbers are all greater than nothing, or 0, and that negative numbers are all less than nothing, or 0; so that whatever exceeds 0, is expressed by positive numbers, and whatever is less than 0, is expressed by negative numbers. The square roots of negative numbers, therefore, are neither greater nor less than nothing. We cannot say, however, that they are 0; for 0 multiplied by 0 produces 0, and consequently does not give a negative number.

143. Now, since all numbers, which it is possible to conceive, are either greater or less than 0, or are 0 itself, it is evident that we cannot rank the square root of a negative number amongst possible numbers, and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers, which from their nature are impossible. *These numbers are usually called imaginary quantities*, because they exist merely in the imagination.

144. All such expressions, as $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$, &c., are consequently impossible, or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert, that they are neither nothing, nor greater than nothing, nor less than nothing; which necessarily constitutes them imaginary, or impossible.

145. But notwithstanding all this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{-4}$ is meant a number, which multiplied by itself, produces — 4. For this reason also, nothing prevents us from making use of these imaginary numbers, and employing them in calculation.

146. The first idea that occurs on the present subject is, that the square of $\sqrt{-3}$, for example, or the product of $\sqrt{-3}$ by $\sqrt{-3}$,

must be -3; that the product of $\sqrt{-1}$ by $\sqrt{-1}$ is -1; and, in general, that by multiplying $\sqrt{-a}$ by $\sqrt{-a}$, or by taking the square of $\sqrt{-a}$, we obtain $-a$.

147. Now, as $-a$ is equal to $+a$ multiplied by -1, and as the square root of a product is found by multiplying together the roots of its factors, it follows that the root of a multiplied by -1, or $\sqrt{-a}$, is equal to $\sqrt{a}$, multiplied by $\sqrt{-1}$. Now $\sqrt{a}$ is a possible or real number, consequently *the whole impossibility of an imaginary quantity may be always reduced to* $\sqrt{-1}$. For this reason, $\sqrt{-4}$ is equal to $\sqrt{4}$ multiplied by $\sqrt{-1}$, and equal to $2\sqrt{-1}$, on account of $\sqrt{4}$ being equal to 2. For the same reason, $\sqrt{-9}$ is reduced to $\sqrt{9} \times \sqrt{-1}$, or $3\sqrt{-1}$; and $\sqrt{-16}$ is equal to $4\sqrt{-1}$.

148. Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$; and $\sqrt{4}$, or 2, for the value of the product of $\sqrt{-1}$ by $\sqrt{-4}$. We see, therefore, that *two imaginary numbers, multiplied together, produce a real, or possible one.*

But, on the contrary, *a possible number, multiplied by an impossible number, gives always an imaginary product:* thus, $\sqrt{-3}$ by $\sqrt{+5}$ gives $\sqrt{-15}$.

149. It is the same with regard to division; for $\sqrt{a}$ divided by $\sqrt{b}$ making $\sqrt{\frac{a}{b}}$, it is evident that $\sqrt{-4}$ divided by $\sqrt{-1}$ will make $\sqrt{+4}$, or 2; that $\sqrt{+3}$ divided by $\sqrt{-3}$ will give $\sqrt{-1}$; and that 1 divided $\sqrt{-1}$ gives $\sqrt{\frac{+1}{-1}}$, or $\sqrt{-1}$; because 1 is equal to $\sqrt{+1}$.

150. We have before observed, that the square root of any number has always two values, one positive and the other negative; that $\sqrt{4}$, for example, is both $+2$ and -2, and that in general, we must take $-\sqrt{a}$ as well as $+\sqrt{a}$ for the square root of a. This remark applies also to imaginary numbers; *the square root of* $-a$ *is both* $+\sqrt{-a}$ and $-\sqrt{-a}$; *but we must not confound the signs* $+$ *and* $-$, *which are before the radical sign* $\sqrt{}$, *with the sign which comes after it.*

151. It remains for us to remove any doubt which may be entertained concerning the utility of the numbers of which we have been speaking; for those numbers being impossible, it would not be surprising if any one should think them entirely useless, and the subject only of idle speculation. This, however, is not the case. The cal-

culation of imaginary quantities is of the greatest importance: questions frequently arise, of which we cannot immediately say, whether they include any thing real and possible, or not. Now, when the solution of such a question leads to imaginary numbers, we are certain that what is required is impossible.*

CHAPTER XIV.

Of Cubic Numbers.

152. WHEN *a number has been multiplied twice by itself, or, which is the same thing, when the square of a number has been multiplied once more by that number, we obtain a product which is called a cube, or a cubic number.* Thus, the cube of a is $a\ a\ a$, since it is the product obtained by multiplying a by itself, or by a, and that square $a\,a$ again by a.

The cubes of the natural numbers therefore succeed each other in the following order.

Numbers	1	2	3	4	5	6	7	8	9	10
Cubes	1	8	27	64	125	216	343	512	729	1000

153. If we consider the differences of these cubes, as we did those of the squares, by subtracting each cube from that which comes after it, we shall obtain the following series of numbers:

7, 19, 37, 61, 91, 127, 169, 217, 271.

At first we do not observe any regularity in them; but if we take the respective differences of these numbers, we find the following series:

* This is followed in the original by an example intended to illustrate what is here said. It is omitted by the editor, as it implies a degree of acquaintance with the subject which the learner cannot be supposed to possess at this stage of his progress.

12, 18, 24, 30, 36, 42, 48, 54, 60;

in which the terms, it is evident, increase always by 6.

154. After the definition we have given of a cube, it will not be difficult to find the cube of fractional numbers; $\frac{1}{8}$ is the cube of $\frac{1}{2}$; $\frac{1}{27}$ is the cube of $\frac{1}{3}$; and $\frac{8}{27}$ is the cube of $\frac{2}{3}$. In the same manner, we have only to take the cube of the numerator, and that of the denominator separately, and we shall have as the cube of $\frac{3}{4}$, for instance, $\frac{27}{64}$.

155. If it be required *to find the cube of a mixed number, we must first reduce it to a single fraction, and then proceed in the manner that has been described.* To find, for example, the cube of $1\frac{1}{2}$, we must take that of $\frac{3}{2}$, which is $\frac{27}{8}$, or 3 and $\frac{3}{8}$. So the cube of $1\frac{1}{4}$, or of the single fraction $\frac{5}{4}$, is $\frac{125}{64}$, or $1\frac{61}{64}$; and the cube of $3\frac{1}{4}$, or of $\frac{13}{4}$ is $\frac{2197}{64}$, or $34\frac{21}{64}$.

156. Since $a\,a\,a$ is the cube of a, that of $a\,b$ will be $a\,a\,a\,b\,b\,b$; whence we see, that *if a number has two or more factors, we may find its cube by multiplying together the cubes of those factors.* For example, as 12 is equal to 3×4, we multiply the cube of 3, which is 27, by the cube of 4, which is 64, and we obtain 1728, for the cube of 12. Further, the cube of $2\,a$ is $8\,a\,a\,a$, and consequently 8 times greater than the cube of a: and likewise, the cube of $3\,a$ is $27\,a\,a\,a$, that is to say, 27 times greater than the cube of a.

157. *Let us attend here also to the signs* $+$ *and* $-$. It is evident that the cube of a positive number $+\,a$ must also be positive, that is $+\,a\,a\,a$. But if it be required to cube a negative number $-\,a$, it is found by first taking the square, which is $+\,a\,a$, and then multiplying, according to the rule, this square by $-\,a$, which gives for the cube required $-\,a\,a\,a$. *In this respect,* therefore, *it is not the same with cubic numbers as with squares, since the latter are always positive: whereas the cube of* $-\,1$ *is* $-\,1$, *that of* $-\,2$ *is* $-\,8$, *that of* $-\,3$ *is* $-\,27$, *and so on.*

CHAPTER XV.

Of Cube Roots, and of Irrational Numbers resulting from them.

158. As we can, in the manner already explained, find the cube of a given number, so, when a number is proposed, we may also reciprocally find a number, which, multiplied twice by itself, will produce that number. The number here sought is called, with relation to the other, *the cube root. So that the cube root of a given number is the number whose cube is equal to that given number.*

159. It is easy therefore to determine the cube root, when the number proposed is a real cube, such as the examples in the last chapter. For we easily perceive that the cube root of 1 is 1; that of 8 is 2; that of 27 is 3; that of 64 is 4, and so on. And in the same manner, the cube root of — 27 is — 3; and that of — 125 is — 5.

Further, if the proposed number be a fraction, as $\frac{8}{27}$, the cube root of it must be $\frac{2}{3}$; and that of $\frac{64}{343}$ is $\frac{4}{7}$. Lastly, the cube root of a mixed number $2\frac{10}{27}$ must be $\frac{4}{3}$, or $1\frac{1}{3}$: because $2\frac{10}{27}$ is equal to $\frac{64}{27}$.

160. But if the proposed number be not a cube, its cube root cannot be expressed either in integers or in fractional numbers. For example, 43 is not a cubic number; I say therefore, that it is impossible to assign any number, either integer or fractional, whose cube shall be exactly 43. We may however affirm, that the cube root of that number is greater than 3, since the cube of 3 is only 27; and less than 4, because the cube of 4 is 64. We know, therefore, that the cube root required is necessarily contained between the numbers 3 and 4.

161. Since the cube root of 43 is greater than 3, if we add a fraction to 3, it is certain that we may approximate still nearer and nearer to the true value of this root; but we can never assign the number which expresses that value exactly; because the cube of a mixed number can never be perfectly equal to an integer, such as 43. If we were to suppose, for example, $3\frac{1}{2}$ or $\frac{7}{2}$ to be the cube root required, the error would be $\frac{1}{8}$; for the cube of $\frac{7}{2}$ is only $\frac{343}{8}$, or $42\frac{7}{8}$.

162. This therefore shows, that *the cube root of 43 cannot be expressed in any way, either by integers or by fractions.* However,

we have a distinct idea of the magnitude of this root; which induces us to use, in order to represent it, *the sign* $\sqrt[3]{}$, which we place before the proposed number, and which *is read cube root, to distinguish it from the square root, which is often called simply the root*. Thus $\sqrt[3]{43}$ means the cube root of 43, that is to say, the number whose cube is 43, or which, multiplied twice by itself, produces 43.

163. It is evident also, that such expressions cannot belong to rational quantities, and that they rather form a particular species of irrational quantities. They have nothing in common with square roots, and it is not possible to express such a cube root by a square root; as, for example, by $\sqrt{12}$; for the square of $\sqrt{12}$ being 12, its cube will be $12\sqrt{12}$, consequently still irrational, and such cannot be equal to 43.

164. If the proposed number be a real cube, our expressions become rational; $\sqrt[3]{1}$ *is equal to* 1: $\sqrt[3]{8}$ *is equal to* 2; $\sqrt[3]{27}$ *is equal to* 3; *and, generally,* $\sqrt[3]{aaa}$ *is equal to* a.

165. *If it were proposed to multiply one cube root* $\sqrt[3]{a}$ *by another,* $\sqrt[3]{b}$, *the product must be* $\sqrt[3]{ab}$; for we know that the cube root of a product $a\,b$ is found by multiplying together the cube roots of the factors (156). Hence, also, *if we divide* $\sqrt[3]{a}$ *by* $\sqrt[3]{b}$, *the quotient will be* $\sqrt[3]{\frac{a}{b}}$.

166. We further perceive that $2\sqrt[3]{a}$ *is equal to* $\sqrt[3]{8a}$, because 2 is equivalent to $\sqrt[3]{8}$; that $3\sqrt[3]{a}$ is equal to $\sqrt[3]{27a}$, and $b\sqrt[3]{a}$ is equal $\sqrt[3]{abbb}$. So, reciprocally, if the number under the radical sign has a factor which is a cube, we may make it disappear by placing its cube root before the sign. For example, instead of $\sqrt[3]{64a}$ we may write $4\sqrt[3]{a}$; and $5\sqrt[3]{a}$ instead of $\sqrt[3]{125a}$. Hence $\sqrt[3]{16}$ *is equal to* $2\sqrt[3]{2}$, because 16 is equal to 8×2.

167. When a number proposed is negative, its cube root is not subject to the same difficulties that occurred in treating of square roots. For, since the cubes of negative numbers are negative, it follows that the cube roots of negative numbers are only negative. Thus $\sqrt[3]{-8}$ is equal to -2, and $\sqrt[3]{-27}$ to -3. It follows also,

that $\sqrt[3]{-12}$ is the same as $-\sqrt[3]{12}$, and that $\sqrt[3]{-a}$ may be expressed by $-\sqrt[3]{a}$. Whence we see that the sign —, when it is found after the sign of the cube root, might also have been placed before it. We are not, therefore, here led to impossible, or imaginary numbers, as we were in considering the square roots of negative numbers.

CHAPTER XVI.

Of Powers in general.

168. The *product which we obtain by multiplying a number several times by itself, is called a power.* Thus, a square which arises from the multiplication of a number by itself, and a cube which we obtain by multiplying a number twice by itself, are powers. *We say also in the former case, that the number is raised to the second degree, or to the second power; and in the latter, that the number is raised to the third degree, or to the third power.*

169. We distinguish these powers from one another by the number of times that the given number has been used as a factor. For example, a square is called the second power, because a certain given number has been used twice as a factor; and if a number has been used thrice as a factor, we call the product the third power, which therefore means the same as the cube. Multiply a number by itself till you have used it four times as a factor, and you will have its fourth power, or what is commonly called the *bi-quadrate.* From what has been said it will be easy to understand what is meant by the fifth, sixth, seventh, &c. power of a number. I only add, that the names of these powers, after the fourth degree, cease to have any other but these numeral distinctions.

170. To illustrate this still further, we may observe, in the first place, that *the powers of* 1 *remain always the same;* because whatever number of times we multiply 1 by itself, the product is found to be always 1. We shall, therefore, begin by representing the powers of 2 and of 3. They succeed in the following order:

Powers.	Of the number 2.	Of the number 3.
I.	2	3
II.	4	9
III.	8	27
IV.	16	81
V.	32	243
VI.	64	729
VII.	128	2187
VIII.	256	6561
IX.	512	19683
X.	1024	59049
XI.	2048	177147
XII.	4096	531441
XIII.	8192	1594323
XIV.	16384	4782969
XV.	32768	14348907
XVI.	65536	43046721
XVII.	131072	129140163
XVIII.	262144	387420489

But the powers of the number 10 are the most remarkable; for on these powers the system of our arithmetic is founded. A few of them arranged in order, and beginning with the first power, are as follows:

I.	II.	III.	IV.	V.	VI.	
10,	100,	1000,	10000,	100000,	1000000,	&c.

171. In order to illustrate this subject, and to consider it in a more general manner, we may observe, that the powers of any number, a, succeed each other in the following order.

I.	II.	III.	IV.	V.	VI.	
a,	$a\,a$,	$a\,a\,a$,	$a\,a\,a\,a$,	$a\,a\,a\,a\,a$,	$a\,a\,a\,a\,a\,a$,	&c.

But we soon feel the inconvenience attending this manner of writing powers, which consists in the necessity of repeating the same letter very often, to express high powers; and the reader also would have no less trouble, if he were obliged to count all the letters to know what power is intended to be represented. The hundredth power, for example, could not be conveniently written in this manner; and it would be still more difficult to read it.

172. To avoid this inconvenience, a much more commodious method of expressing such powers has been devised, which from its extensive use deserves to be carefully explained; *viz.* To express,

for example, the hundredth power, we simply write the number 100 above the number whose hundredth power we would express, and a little towards the right hand; thus a^{100} means a raised to 100, and *represents the hundredth power of* a. It must be observed, that *the name exponent is given to the number written above that whose power or degree it represents, and which in the present instance is* 100.

173. In the same manner, a^2 signifies a raised to 2, or the second power of a, which we represent sometimes also by $a\,a$, because both these expressions are written and understood with equal facility. But to express the cube, or the third power $a\,a\,a$, we write a^3 according to the rule, that we may occupy less room. So a^4 signifies the fourth, a^5 the fifth, and a^6 the sixth power of a.

174. In a word, all the powers of a will be represented by a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, &c. Whence we see that in this manner we might very properly have written a^1 instead of a for the first term, to show the order of the series more clearly. In fact a^1 *is no more than* a, *as this unit shows that the letter* a *is to be written only once.* Such a series of powers is called also a geometrical progression, because each term is greater by one than the preceding.

175. As in this series of powers each term is found by multiplying the preceding term by a, which increases the exponent by 1; so when any term is given, we may also find the preceding one, if we divide by a, because this diminishes the exponent by 1. This shows that the term which precedes the first term a^1 must necessarily be $\frac{a}{a}$, or 1; now, if we proceed according to the exponents, we immediately conclude, that the term which precedes the first must be a^0. Hence we deduce this remarkable property; that a^0 *is constantly equal to* 1, *however great or small the value of the number* a *may be,* and even when a is nothing; that is to say, a^0 is equal to 1.

176. We may continue our series of powers in a retrograde order, and that in two different ways; first, by dividing always by a, and secondly by diminishing the exponent by unity. And it is evident, that whether we follow the one or the other, the terms are still perfectly equal. This decreasing series is represented, in both forms, in the following table, which must be read backwards, or from right to left.

	$\frac{1}{aaaaaa}$	$\frac{1}{aaaaa}$	$\frac{1}{aaaa}$	$\frac{1}{aaa}$	$\frac{1}{aa}$	$\frac{1}{a}$	1	a
1.	$\frac{1}{a^6}$	$\frac{1}{a^5}$	$\frac{1}{a^4}$	$\frac{1}{a^3}$	$\frac{1}{a^2}$	$\frac{1}{a^1}$		
2.	a^{-6}	a^{-5}	a^{-4}	a^{-3}	a^{-2}	a^{-1}	a^0	a^1

177. We are thus brought to understand the nature of powers, whose exponents are negative, and are enabled to assign the precise value of these powers. From what has been said, it appears that,

$$\left.\begin{array}{l} a^0 \\ a^{-1} \\ a^{-2} \\ a^{-3} \\ a^{-4} \end{array}\right\} \text{is equal to} \left\{\begin{array}{l} 1;\ \text{then} \\ \frac{1}{a}; \\ \frac{1}{aa},\ \text{or}\ \frac{1}{a^2}; \\ \frac{1}{a^3}; \\ \frac{1}{a^4},\ \text{\&c.} \end{array}\right.$$

178. It will be easy, from the foregoing notation, to find *the powers of a product*, a b. *They must evidently be* a b, *or* $a^1\,b^1$, $a^2\,b^2$, $a^3\,b^3$, $a^4\,b^4$, $a^5\,b^5$, *&c. And the powers of fractions will be found in the same manner; for example, those of* $\frac{a}{b}$ *are,*

$$\frac{a^1}{b^1},\ \frac{a^2}{b^2},\ \frac{a^3}{b^3},\ \frac{a^4}{b^4},\ \frac{a^5}{b^5},\ \frac{a^6}{b^6},\ \frac{a^7}{b^7}\ \text{\&c.}$$

179. Lastly, we have to consider the powers of negative numbers. Suppose the given number to be $-a$; its powers will form the following series:

$$-a,\ +aa,\ -a^3,\ +a^4,\ -a^5,\ +a^6,\ \text{\&c.}$$

We may observe that those powers only become negative, whose exponents are odd numbers, and that, on the contrary, all the powers, which have an even number for the exponent, are positive. So that, the third, fifth, seventh, ninth, &c. powers have each the sign $-$; and the second, fourth, sixth, eighth, &c. powers are affected with the sign $+$.

CHAPTER XVII.

Of the Calculation of Powers.

180. We have nothing in particular to observe with regard to the addition and subtraction of powers; for we only represent these operations by means of the signs + and —, when the powers are different. For example, $a^3 + a^2$ *is the sum of the second and third powers of* a; *and* $a^5 - a^4$ *is what remains when we subtract the fourth power of* a *from the fifth; and neither of these results can be abridged.* When we have powers of the same kind, or degree, it is evidently unnecessary to connect them by signs; $a^2 + a^2$ makes $2a^2$, &c.

181. But in the multiplication of powers, several things require attention.

First, when it is required to multiply any power of a by a, we obtain the succeeding power, that is to say, the power whose exponent is greater by one unit. Thus a^2, multiplied by a, produces a^3; and a^3 multiplied by a, produces a^4. And, in the same manner, when it is required to multiply by a the powers of that number which have negative exponents, we must add 1 to the exponent. Thus, a^{-1} *multiplied by* a *produces* a^0 *or* 1; which is made more evident by considering that a^{-1} is equal to $\frac{1}{a}$, and that the product of $\frac{1}{a}$ by a being $\frac{a}{a}$, it is consequently equal to 1. Likewise a^{-2} multiplied by a produces a^{-1}, or $\frac{1}{a}$; and a^{-10}, multiplied by a, gives a^{-9}, and so on.

182. Next, if it be required to multiply a power of a by $a\,a$, or the second power, I say that the exponent becomes greater by 2. Thus, the product of a^2 by a^2 is a^4; that of a^2 by a^3 is a^5; that of a^4 by a^2 is a^6; and, more generally, a^n *multiplied by* a^2 *makes* a^{n+2}. *With regard to negative exponents, we shall have* a^1, *or* a, *for the product of* a^{-1} *by* a^2; for a^{-1} being equal to $\frac{1}{a}$, it is the same as if we had divided $a\,a$ by a; consequently the product required is $\frac{a\,a}{a}$, or a. So a^{-2}, *multiplied by* a^2, *produces* a^0 *or* 1; *and* a^{-3}, *multiplied by* a^2, *produces* a^{-1}.

183. It is no less evident that to multiply any power of a by a^3, we must increase its exponent by three units; and that consequently the product of a^n by a^3 is a^{n+3}. And *whenever it is required to multiply together two powers of* a, *the product will be also a power of* a, *and a power whose exponent will be the sum of the exponents of the two given powers*. For example, a^4, multiplied by a^5, will make a^9, and a^{12}, multiplied by a^7, will produce a^{19}, &c.

184. From these considerations we may easily determine the highest powers. To find, for instance, the twenty-fourth power of 2, I multiply the twelfth power by the twelfth power, because 2^{24} is equal to $2^{12} \times 2^{12}$. Now we have already seen that 2^{12} is 4096; I say, therefore, that the number 16777216, or the product of 4096 by 4096, expresses the power required, 2^{24}.

185. Let us proceed to division. We shall remark in the first place, that *to divide a power of* a *by* a *we must subtract* 1 *from the exponent, or diminish it by unity*. Thus a^5, divided by a, gives a^4; a^0, or 1, divided by a, is equal to a^{-1} or $\frac{1}{a}$; a^{-3}, divided by a, gives a^{-4}.

186. If we have to divide a given power of a by a^2, we must diminish the exponent by 2; and if by a^3, we must subtract three units from the exponent of the power proposed. So, *in general, whatever power of* a *it is required to divide by another power of* a, *the rule is always to subtract the exponent of the second from the exponent of the first of these powers*. Thus a^{15}, divided by a^7, will give a^8, a^6, divided by a^7, will give a^{-1}; and a^{-3}, divided by a^4, will give a^{-7}.

187. From what has been said above, it is easy to understand the method of finding the powers of powers, this being done by multiplication. When we seek, for example, the square, or the second power of a^3, we find a^6; and in the same manner we find a^{12} for the third power of the cube of a^4. *To obtain the square of a power, we have only to double its exponent; for its cube, we must triple the exponent; and so on*. The square of a^n is a^{2n}; the cube of a^n is a^{3n}; the seventh power of a^n is a^{7n}, &c.

188. The square of a^2, or the square of the square of a, being a^4, we see why the fourth power is called the *bi-quadrate*. The square of a^3 is a^6; the sixth power has therefore received the name of the *square-cubed*.

Lastly, the cube of a^3 being a^9 we call the ninth power the *cubo-cube*. No other denominations of this kind have been introduced for powers, and indeed the two last are very little used.

CHAPTER XVIII.

Of Roots with relation to Powers in general.

189. Since the square root of a given number is a number, whose square is equal to that given number; and since the cube root of a given number is a number whose cube is equal to that given number; it follows that any number whatever being given, we may always indicate such roots of it, that their fourth, or their fifth, or any other power, may be equal to the given number. To distinguish these different kinds of roots better, we shall call the square root the *second root*; and the cube root the *third root*; because, according to this denomination, we may call the *fourth root*, that whose biquadrate is equal to a given number; and the *fifth root*, that whose fifth power is equal to a given number, &c.

190. As the square, or second root, is marked by the sign $\sqrt{}$, and the cubic or third root by the sign $\sqrt[3]{}$, so the fourth root is represented by the sign $\sqrt[4]{}$; the fifth root by the sign $\sqrt[5]{}$; and so on; it is evident that according to this mode of expression, the sign of the square root, ought to be $\sqrt[2]{}$. But as of all roots this occurs most frequently, it has been agreed, for the sake of brevity, to omit the number 2 in the sign of this root. So that when a radical sign has no number prefixed, this always shows that the square root is to be understood.

191. To explain this matter still further, we shall here exhibit the different roots of the number a, with their respective values:

$$\left.\begin{array}{l}\sqrt{a}\\ \sqrt[3]{a}\\ \sqrt[4]{a}\\ \sqrt[5]{a}\\ \sqrt[6]{a}\end{array}\right\}\text{is the}\left\{\begin{array}{l}\text{2d}\\ \text{3d}\\ \text{4th}\\ \text{5th}\\ \text{6th}\end{array}\right\}\text{root of}\left\{\begin{array}{l}a,\\ a,\\ a,\\ a,\\ a,\text{ and so on.}\end{array}\right.$$

So that conversely;

$$\left.\begin{array}{l}\text{The 2d}\\ \text{The 3d}\\ \text{The 4th}\\ \text{The 5th}\\ \text{The 6th}\end{array}\right\}\text{power of}\left\{\begin{array}{l}\sqrt{a}\\ \sqrt[3]{a}\\ \sqrt[4]{a}\\ \sqrt[5]{a}\\ \sqrt[6]{a}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}a,\\ a,\\ a,\\ a,\\ a,\text{ and so on.}\end{array}\right.$$

192. Whether the number a therefore be great or small, we know what value to affix to all these roots of different degrees.

It must be remarked also, that if we substitute unity for a, all those roots remain constantly 1; because all the powers of 1 have unity for their value. If the number a be greater than 1, all its roots will also exceed unity. Lastly, if that number be less than 1, all its roots will also be less than unity.

193. When the number a is positive, we know from what was before said of the square and cube roots, that all the other roots may also be determined, and will be real and possible numbers.

But if the number a is negative, its second, fourth, sixth, and all the even roots, become impossible, or imaginary numbers; because *all the even powers, whether of positive or of negative numbers, are affected with the sign +. Whereas the third, fifth, seventh, and all odd roots, become negative,* but rational; because the odd powers of negative numbers, are also negative.

194. We have here also an inexhaustible source of new kinds of surd, or irrational quantities; for whenever the number a is not actually such a power, as some one of the foregoing indices represents, or seems to require, it is impossible to express that root either in whole numbers or in fractions; and consequently it must be classed among the numbers which are called irrational.

CHAPTER XIX.

Of the Method of representing Irrational Numbers by Fractional Exponents.

195. We have shown in the preceding chapter, that the square of any power is found by doubling the exponent of that power, and that in general the square, or the second power of a^n, is a^{2n}. The converse follows, namely, that *the square root of the power* a^{2n} *is* a^n, and that *it is found by taking half the exponent of that power, or dividing it by* 2.

196. Thus the square root of a^2 is a^1; that of a^4 is a^2; that of a^6 is a^3; and so on. And as this is general, the square root of a^3 must necessarily be $a^{\frac{3}{2}}$, and that of a^5, $a^{\frac{5}{2}}$. Consequently we shall have in the same manner $a^{\frac{1}{2}}$ for the square root of a^1; whence we see that $a^{\frac{1}{2}}$ *is equal to* $\sqrt{a}$; and this new method of representing the square root demands particular attention.

197. We have also shown that to find the cube of a power as a^n, we must multiply its exponent by 3, and that consequently the cube is a^{3n}.

So conversely, when it is required to find the third or cube root of the power a^{3n}, we have only to divide the exponent by 3, and may with certainty conclude, that the root required is a^n. Consequently a^1, or a, is the cube root of a^3; a^2 is that of a^6; a^3 is that of a^9; and so on.

198. There is nothing to prevent us from applying the same reasoning to those cases in which the exponent is not divisible by 3, and concluding that the cube root of a^2 is $a^{\frac{2}{3}}$, and that the cube root of a^4 is $a^{\frac{4}{3}}$, or $a^{1\frac{1}{3}}$. Consequently the third, or cube root of a also, or a^1, must be $a^{\frac{1}{3}}$ Whence it appears that $a^{\frac{1}{3}}$ *is equal to* $\sqrt[3]{a}$.

199. It is the same with roots of a higher degree. The fourth root of a will be $a^{\frac{1}{4}}$, which expression has the same value as $\sqrt[4]{a}$. The fifth root of a will be $a^{\frac{1}{5}}$, which is consequently equivalent to $\sqrt[5]{a}$; and the same observation may be extended to all roots of a higher degree.

200. We might, therefore, entirely reject the radical signs at present made use of, and employ in their stead the fractional exponents which we have explained; however, as we have been long accustomed to those signs, and meet with them in all books of algebra, it would be wrong to banish them entirely. But there is sufficient reason also to employ, as is now frequently done, the other method of notation, because it manifestly corresponds with what is to be represented. In fact, we see immediately that $a^{\frac{1}{2}}$ is the square root of a, because we know that the square of $a^{\frac{1}{2}}$, that is to say, $a^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{2}}$, is equal to a^1 or a.

201. What has now been said is sufficient to show how we are to understand all other fractional exponents that may occur. If we have, for example, $a^{\frac{4}{3}}$, this means that we must first take the fourth power of a, and then extract its cube or third root; so that $a^{\frac{4}{3}}$ *is the same as the common expression*, $\sqrt[3]{a^4}$. To find the value of $a^{\frac{3}{4}}$, we must first take the cube, or the third power of a, which is a^3, and then extract the fourth root of that power; so that $a^{\frac{3}{4}}$ is the same as $\sqrt[4]{a^3}$. Also $a^{\frac{4}{5}}$ is equal to $\sqrt[5]{a^4}$, &c.

202. When the fraction which represents the exponent exceeds unity, we may express the value of the given quantity in another way. Suppose it to be $a^{\frac{5}{2}}$; this quantity is equivalent to $a^{2\frac{1}{2}}$, which is the product of a^2 by $a^{\frac{1}{2}}$. Now $a^{\frac{1}{2}}$ being equal to $\sqrt{a}$, it is evident that $a^{\frac{5}{2}}$ is equal to $a^2\sqrt{a}$. So $a^{\frac{10}{3}}$, or $a^{3\frac{1}{2}}$, is equal to $a^3\sqrt[3]{a}$; and $a^{\frac{15}{4}}$, that is $a^{3\frac{3}{4}}$, expresses $a^3\sqrt[4]{a^3}$. These examples are sufficient to illustrate the great utility of fractional exponents.

203. Their use extends also to fractional numbers: let there be given $\frac{1}{\sqrt{a}}$, we know that this quantity is equal to $\frac{1}{a^{\frac{1}{2}}}$; now we have seen already that a fraction of the form $\frac{1}{a^n}$ may be expressed by a^{-n}; so instead of $\frac{1}{\sqrt{a}}$ we may use the expression $a^{-\frac{1}{2}}$. In the same manner, $\frac{1}{\sqrt[3]{a}}$ *is equal to* $a^{-\frac{1}{3}}$. Again, let the quantity $\frac{a^2}{\sqrt[4]{a^3}}$ be

proposed; let it be transformed into this, $\frac{a^2}{a^{\frac{3}{4}}}$, which is the product of a^2 by $a^{-\frac{3}{4}}$; now this product is equivalent to $a^{\frac{5}{4}}$, or to $a^{1\frac{1}{4}}$, or lastly to $a\sqrt[4]{a}$. Practice will render similar reductions easy.

204. We shall observe, in the last place, that each root may be represented in a variety of ways. For $\sqrt{a}$ being the same as $a^{\frac{1}{2}}$, and $\frac{1}{2}$ being transformable into all these fractions, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, &c., it is evident that $\sqrt{a}$ *is equal to* $\sqrt[4]{a^2}$ and to $\sqrt[6]{a^3}$ and to $\sqrt[8]{a^4}$, and so on. In the same manner $\sqrt[3]{a}$, which is equal to $a^{\frac{1}{3}}$, will be equal to $\sqrt[6]{a^2}$, and to $\sqrt[9]{a^3}$, and to $\sqrt[12]{a^4}$. And we see also, that the number a, or a^1, might be represented by the following radical expressions:

$$\sqrt[2]{a^2},\ \sqrt[3]{a^3},\ \sqrt[4]{a^4},\ \sqrt[5]{a^5},\ \&c.$$

205. This property is of great use in multiplication and division: for if we have, for example, to multiply $\sqrt[2]{a}$ by $\sqrt[3]{a}$, we write $\sqrt[6]{a^3}$ for $\sqrt[2]{a}$ and $\sqrt[6]{a^2}$ instead of $\sqrt[3]{a}$; in this manner we obtain the same radical sign for both, and the multiplication being now performed, gives the product $\sqrt[6]{a^5}$. The same result is deduced from $a^{\frac{1}{2}+\frac{1}{3}}$, the product of $a^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{3}}$; for $\frac{1}{2}+\frac{1}{3}$ is $\frac{5}{6}$, and consequently the product required is $a^{\frac{5}{6}}$, or $\sqrt[6]{a^5}$.

If it were required to divide $\sqrt[2]{a}$, *or* $a^{\frac{1}{2}}$, *by* $\sqrt[3]{a}$ *or* $a^{\frac{1}{3}}$, *we should have for the quotient* $a^{\frac{1}{2}-\frac{1}{3}}$, *or* $a^{\frac{3}{6}-\frac{2}{6}}$, *that is to say*, $a^{\frac{1}{6}}$, *or* $\sqrt[6]{a}$.

CHAPTER XX.

Of the different Methods of Calculation, and of their mutual Connexion.

206. Hitherto we have only explained the different methods of calculation: addition, subtraction, multiplication, and division;

the involution of powers, and the extraction of roots. It will not be improper, therefore, in this place, to trace back the origin of these different methods, and to explain the connexion which subsists among them; in order that we may satisfy ourselves whether it be possible or not for other operations of the same kind to exist. This inquiry will throw new light on the subjects which we have considered.

In prosecuting this design, we shall make use of a new character, which may be employed instead of the expression that has been so often repeated, *is equal to*; this sign is $=$, and is read *is equal to*. Thus, when I write $a = b$, this means that a is equal to b; so, for example $3 + 5 = 15$.

207. The first mode of calculation, which presents itself to the mind, is undoubtedly addition, by which we add two numbers together and find their sum. Let a and b then be the two given numbers, and let their sum be expressed by the letter c, we shall have $a + b = c$. So that when we know the two numbers a and b, addition teaches us to find the number c.

208. Preserving this comparison $a + b = c$, let us reverse the question by asking, how we are to find the number b, when we know the numbers a and c.

It is required therefore to know what number must be added to a, in order that the sum may be the number c. Suppose, for example, $a = 3$ and $c = 8$; so that we must have $3 + b = 8$; b will evidently be found by subtracting 3 from 8. So, in general, to find b, we must subtract a from c, whence arises $b = c - a$; for by adding a to both sides again, we have $b + a = c - a + a$, that is to say $= c$, as we supposed.

Such then is the origin of subtraction.

209. Subtraction therefore takes place, when we invert the question which gives rise to addition. Now the number which it is required to subtract may happen to be greater than that from which it is to be subtracted; as, for example, if it were required to subtract 9 from 5: this instance therefore furnishes us with the idea of a new kind of numbers, which we call negative numbers, because $5 - 9 = -4$.

210. When several numbers are to be added together which are all equal, their sum is found by multiplication, and is called a product. Thus ab means the product arising from the multiplication of a by b, or from the addition of a number a to itself b number of

times. If we represent this product by the letter c, we shall have $a\,b = c$; and multiplication teaches us how to determine the number c, when the numbers a and b are known.

211. Let us now propose the following question: the numbers a and c being known, to find the number b. Suppose, for example, $a = 3$ and $c = 15$, so that $3\,b = 15$, we ask by what number 3 must be multiplied, in order that the product may be 15: for the question proposed is reduced to this. Now this is division: the number required is found by dividing 15 by 3; and therefore, in general, the number b is found by dividing c by a; from which results the equation $b = \frac{c}{a}$.

212. Now, as it frequently happens that the number c cannot be really divided by the number a, while the letter b must however have a determinate value, another new kind of numbers presents itself; these are fractions. For example, supposing $a = 4$, $c = 3$, so that $4\,b = 3$, it is evident that b cannot be an integer, but a fraction, and that we shall have $b = \frac{3}{4}$.

213. We have seen that multiplication arises from addition, that is to say, from the addition of several equal quantities. If we now proceed further, we shall perceive that from the multiplication of several equal quantities together powers are derived. Those powers are represented in a general manner by the expression a^b, which signifies that the number a must be multiplied as many times by itself, as is denoted by the number b. And we know from what has been already said, that in the present instance a is called the root, b the exponent, and a^b the power.

214. Further, if we represent this power also by the letter c, we have $a^b = c$, an equation in which three letters, a, b, c, are found. Now we have shown in treating of powers, how to find the power itself, that is, the letter c, when a root a and its exponent b are given. Suppose, for example, $a = 5$, and $b = 3$, so that $c = 5^3$; it is evident that we must take the third power of 5, which is 125, and that thus $c = 125$.

215. We have seen how to determine the power c, by means of the root a and the exponent b; but if we wish to reverse the question, we shall find that this may be done in two ways, and that there are two different cases to be considered: for if two of these three numbers a, b, c, were given, and it were required to find the third, we should immediately perceive that this question admits of three

different suppositions, and consequently three solutions. We have considered the case in which a and b were the numbers given, we may therefore suppose further that c and a, or c and b are known, and that it is required to determine the third letter. Let us point out, therefore, before we proceed any further, a very essential distinction between involution and the two operations which lead to it. When in addition we reversed the question, it could be done only in one way; it was a matter of indifference whether we took c and a, or c and b for the given numbers, because we might indifferently write $a + b$, or $b + a$. It was the same with multiplication; we could at pleasure take the letters a and b for each other, the equation $a\,b = c$ being exactly the same as $b\,a = c$.

In the calculation of powers, on the contrary, the same thing does not take place, and we can by no means write b^a instead of a^b. A single example will be sufficient to illustrate this: let $a = 5$, and $b = 3$; we have $a^b = 5^3 = 125$. But $b^a = 3^5 = 243$: two very different results.

SECTION II.

OF THE DIFFERENT METHODS OF CALCULATION APPLIED TO COMPOUND QUANTITIES.

CHAPTER I.

Of the Addition of Compound Quantities.

Article 216. When two or more expressions, consisting of several terms, are to be added together, the operation is frequently represented merely by signs, placing each expression between two parentheses, and connecting it with the rest by means of the sign $+$. If it be required, for example, to add the expressions $a + b + c$ and $d + e + f$, we represent the sum thus:

$$(a + b + c) + (d + e + f).$$

217. It is evident that this is not to perform addition, but only to represent it. We see at the same time, however, that in order to perform it actually, we have only to leave out the parentheses; for as the number $d + e + f$ is to be added to the other, we know that this is done by joining to it first $+ d$, then $+ e$, and then $+ f$; which therefore gives the sum

$$a + b + c + d + e + f.$$

The same method is to be observed, if any of the terms are affected with the sign $-$; they must be joined in the same way, by means of their proper sign.

218. To make this more evident, we shall consider an example in pure numbers. It is proposed to add the expression $15 - 6$ to $12 - 8$. If we begin by adding 15, we shall have $12 - 8 + 15$; now this was adding too much, since we had only to add $15 - 6$, and it is evident that 6 is the number which we have added too

much. Let us, therefore, take this 6 away by writing it with the negative sign, and we shall have the true sum,

$$12 - 8 + 15 - 6,$$

which shows that the *sums are found by writing all the terms, each with its proper sign.*

219. If it were required therefore to add the expression $d - e - f$ to $a - b + c$, we should express the sum thus:

$$a - b + c + d - e - f,$$

remarking, however, that it is of no consequence in what order we write these terms. Their place may be changed at pleasure, provided their signs be preserved. This sum might, for example, be written thus:

$$c - e + a - f + d - b.$$

220. It frequently happens that the sums represented in this manner may be considerably abridged, as when two or more terms destroy each other; for example, if we find in the same sum the terms $+a - a$, or $3a - 4a + a$: or when two or more terms may be reduced to one. Examples of this second reduction:

$$3a + 2a = 5a;\ 7b - 3b = +4b;$$
$$-6c + 10c = +4c;$$
$$5a - 8a = -3a;\ -7b + b = -6b;$$
$$-3c - 4c = -7c;$$
$$2a - 5a + a = -2a;\ -3b - 5b + 2b = -6b.$$

Whenever two or more terms, therefore, *are entirely the same with regard to letters*, their sum may be abridged; but those cases must not be confounded with such as these, $2aa + 3a$, or $2b^3 - b^4$, which admit of no abridgment.

221. Let us consider some more examples of reduction; the following will lead us immediately to an important truth. Suppose it were required to add together the expressions $a + b$ and $a - b$; our rule gives $a + b + a - b$; now $a + a = 2a$ and $b - b = 0$; the sum then is $2a$: consequently if we add together the sum of two numbers $(a + b)$ and their difference $(a - b,)$ we obtain the double of the greater of those two numbers.

Further examples:

$$\begin{array}{r|r} 3a - 2b - c & a^3 - 2aab + 2abb \\ 5b - 6c + a & -aab + 2abb - b^3 \\ \hline 4a + 3b - 7c & a^3 - 3aab + 4abb - b^3. \end{array}$$

CHAPTER II.

Of the Subtraction of Compound Quantities.

222. If we wish merely to represent subtraction, we inclose each expression within two parentheses, connecting, by the sign —, the expression to be subtracted with that from which it is to be taken.

When we subtract, for example, the expression $d - e + f$ from the expression $a - b + c$, we write the remainder thus:

$$(a - b + c) - (d - e + f);$$

and this method of representing it sufficiently shows, which of the two expressions is to be subtracted from the other.

223. But if we wish to perform the subtraction, we must observe, first, that when we subtract a positive quantity $+ b$ from another quantity a, we obtain $a - b$; and secondly, when we subtract a negative quantity $- b$ from a, we obtain $a + b$; because to free a person from a debt is the same as to give him something.

224. Suppose, now, it were required to subtract the expression $b - d$ from the expression $a - c$, we first take away b; which gives $a - c - b$. Now this is taking too much away by the quantity d, since we had to subtract only $b - d$; we must therefore restore the value of d, and we shall then have

$$a - c - b + d;$$

whence it is evident, that *the terms of the expression to be subtracted must have their signs changed, and be joined, with the contrary signs, to the terms of the other expression.*

225. It is easy, therefore, by means of this rule, to perform subtraction, since we have only to write the expression from which we are to subtract, such as it is, and join the other to it without any change beside that of the signs. Thus, in the first example, where it was required to subtract the expression $d - e + f$ from $a - b + c$, we obtain $a - b + c - d + e - f$.

An example in numbers will render this still more clear. If we subtract $6 - 2 + 4$ from $9 - 3 + 2$, we evidently obtain

$$9 - 3 + 2 - 6 + 2 - 4;$$

for $9 - 3 + 2 = 8$; also, $6 - 2 + 4 = 8$; now $8 - 8 = 0$.

226. Subtraction being therefore subject to no difficulty, we have only to remark, that, if there are found in the remainder two or more terms which are entirely similar with regard to the letters, that remainder may be reduced to an abridged form, by the same rules which we have given in addition.

227. Suppose we have to subtract from $a + b$, or from the sum of two quantities, their difference $a - b$, we shall then have

$$a + b - a + b;$$

now $a - a = 0$, and $b + b = 2\,b$; the remainder sought is therefore $2\,b$, that is to say, the double of the less of the two quantities.

228. The following examples will supply the place of further illustrations.

$aa + ab + bb$	$3a - 4b + 5c$	$a^3 + 3aab + 3abb + b^3$	$\sqrt{a} + 2\sqrt{b}$
$bb + ab - aa$	$2b + 4c - 6a$	$a^3 - 3aab + 3abb - b^3$	$\sqrt{a} - 3\sqrt{b}$
$2aa.$	$9a - 6b + c.$	$6aab + 2b^3.$	$+ 5\sqrt{b}.$

CHAPTER III.

Of the Multiplication of Compound Quantities.

229. When it is only required to represent multiplication, we put each of the expressions, that are to be multiplied together, within two parentheses, and join them to each other, sometimes without any sign, and sometimes placing the sign $\times$ between them. For example, to represent the product of the two expressions $a - b + c$ and $d - e + f$, when multiplied together, we write

$$(a - b + c) \times (d - e + f.)$$

This method of expressing products is much used, because it immediately shows the factors of which they are composed.

230. But to show how multiplication is to be actually performed, we may remark, in the first place, that in order to multiply, for example, a quantity such as $a - b + c$, by 2, each term of it is separately multiplied by that number; so that the product is

$$2\,a - 2\,b + 2\,c.$$

Now the same thing takes place with regard to all other numbers. If d were the number, by which it is required to multiply the same expression, we should obtain

$$a d - b d + c d.$$

231. We supposed d to be a positive number; but if the factor were a negative number, as $-e$, the rule heretofore given must be applied; namely, that *two contrary signs, multiplied together, produce —, and that two like signs give +.*

We shall accordingly have

$$-ae + be - ce.$$

232. To show how a quantity, A, is to be multiplied by a compound quantity, $d - e$; let us consider an example in common numbers, supposing that A is to be multiplied by $7 - 3$. Now it is evident, that we are here required to take the quadruple of A; for if we first take A seven times, it will then be necessary to subtract $3A$ from that product.

In general, therefore, if it be required to multiply by $d - e$, we multiply the quantity A first by d and then by e, and subtract this last product from the first; whence results $dA - eA$.

Suppose now $A = a - b$, and that this is the quantity to be multiplied by $d - e$; we shall have

$$\begin{array}{l} dA = ad - bd \\ eA = ae - be \\ \hline \end{array}$$

whence the product required $= ad - bd - ae + be$.

233. Since we know therefore the product $(a - b) \times (d - e)$, and cannot doubt of its accuracy, we shall exhibit the same example of multiplication under the following form:

$$\begin{array}{c} a - b \\ d - e \\ \hline ad - bd - ae + be \end{array}$$

This shows, that we must *multiply each term of the upper expression by each term of the lower*, and that, with regard to the signs, we must strictly observe the rule before given; a rule which this would completely confirm, if it admitted of the least doubt.

234. It will be easy, according to this rule, to perform the following example, which is, to multiply $a + b$ by $a - b$;

$$\begin{array}{l} a + b \\ a - b \\ \hline aa + ab \\ \quad\; - ab - bb. \\ \hline \text{Product } aa - bb. \end{array}$$

235. Now we may substitute, for a and b, any determinate numbers; so that the above example will furnish the following theorem; viz. *The product of the sum of two numbers, multiplied by their difference, is equal to the difference of the squares of those numbers.* This theorem may be expressed thus:

$$(a + b) \times (a - b) = aa - bb.$$

And from this another theorem may be derived; namely, *The difference of two square numbers is always a product, and divisible both by the sum and by the difference of the roots of those two squares.*

226. Let us now perform some other examples:

$$\begin{array}{l} \text{I.)}\ 2a - 3 \\ \quad\; a + 2 \\ \hline 2aa - 3a \\ \quad\; + 4a - 6 \\ \hline 2aa + a - 6. \end{array}$$

$$\begin{array}{l} \text{II.)}\ 4aa - 6a + 9 \\ \quad\; 2a + 3 \\ \hline 8a^3 - 12aa + 18a \\ \quad\; + 12aa - 18a + 27 \\ \hline 8a^3 + 27 \end{array}$$

$$\begin{array}{l} \text{III.)}\ 3aa - 2ab - bb \\ \quad\; 2a - 4b \\ \hline 6a^3 - 4aab - 2abb \\ \quad\; - 12aab + 8abb + 4b^3 \\ \hline 6a^3 - 16aab + 6abb + 4b^3 \end{array}$$

$$\begin{array}{l} \text{IV.)}\ aa + 2ab + 2bb \\ \quad\ \ aa - 2ab + 2bb \\ \hline a^4 + 2a^3b + 2aabb \\ \qquad - 2a^3b - 4aabb - 4ab^3 \\ \qquad\qquad + 2aabb + 4ab^3 + 4b^4 \\ \hline \qquad a^4 + 4b^4. \end{array}$$

$$\begin{array}{l} \text{V.)}\ 2aa - 3ab - 4bb \\ \quad\ \ 3aa - 4ab + bb \\ \hline 6a^4 - 9a^3b - 12aabb \\ \qquad - 4a^3b + 6aabb + 8ab^3 \\ \qquad\qquad + 2aabb - 3ab^3 - 4b^4 \\ \hline 6a^4 - 13a^3b - 4aabb + 5ab^3 - 4b^4 \end{array}$$

$$\begin{array}{l} \text{VI.)}\ aa + bb + cc - ab - ac - bc \\ \qquad a + b + c \\ \hline a^3 + abb + acc - aab - aac - abc \\ \quad aab + b^3 + bcc - abb - abc - bbc \\ \qquad aac + bbc + c^3 - abc - acc - bcc \\ \hline \qquad a^3 - 3abc + b^3 + c^3 \end{array}$$

237. *When we have more than two quantities to multiply together, it will easily be understood that after having multiplied two of them together, we must then multiply that product by one of those which remain, and so on. It is indifferent what order is observed in these multiplications.*

Let it be proposed, for example, to find the value, or product, of the four following factors, *viz.*

I.	II.	III.	IV.
$(a + b)$	$(aa + ab + bb)$	$(a - b)$	$(aa - ab + bb)$.

We will first multiply the factors I. and II.

$$\begin{array}{l} \text{II.}\ aa + ab + bb \\ \text{I.}\ \quad a + b \\ \hline a^3 + aab + abb \\ \quad + aab + abb + b^3 \\ \hline \end{array}$$

$$\text{I. II.} = a^3 + 2aab + 2abb + b^3.$$

Next let us multiply the factors III. and IV.

$$\begin{array}{rl} \text{IV.} & aa - ab + bb \\ \text{III.} & a - b \\ \hline & a^3 - aab + abb \\ & \quad - aab + abb - b^3 \\ \hline \end{array}$$

$$\text{III. IV.} = a^3 - 2aab + 2abb - b^3.$$

It remains now to multiply the first product I. II. by this second product III. IV.:

$$\begin{array}{l} a^3 + 2aab + 2abb + b^3 \quad \text{I. II.} \\ a^3 - 2aab + 2abb - b^3 \quad \text{III. IV.} \\ \hline a^6 + 2a^5b + 2a^4bb + a^3b^3 \\ \quad - 2a^5b - 4a^4bb - 4a^3b^3 - 2aab^4 \\ \qquad 2a^4bb + 4a^3b^3 + 4aab^4 + 2ab^5 \\ \qquad\quad - a^3b^3 - 2aab^4 - 2ab^5 - b^6 \\ \hline a^6 - b^6. \end{array}$$

And this is the product required.

238. Let us resume the same example, but change the order of it, first multiplying the factors I. and III. and then II. and IV. together.

$$\begin{array}{rl} \text{I.} & a + b \\ \text{III.} & a - b \\ \hline & aa + ab \\ & \quad - ab - bb \\ \hline \end{array}$$

$$\text{I. III.} = aa - bb.$$

$$\begin{array}{rl} \text{II.} & aa + ab + bb \\ \text{IV.} & aa - ab + bb \\ \hline & a^4 + a^3b + aabb \\ & \quad - a^3b - aabb - ab^3 \\ & \qquad aabb + ab^3 + b^4 \\ \hline \end{array}$$

$$\text{II. IV.} = a^4 + aabb + b^4.$$

Then multiplying the two products I. III. and II. IV.,

$$\begin{array}{l} \text{II. IV.} = a^4 + aabb + b^4 \\ \text{I. III.} = aa - bb \\ \hline a^6 + a^4bb + aab^4 \\ \quad - a^4b - aab^4 - b^6 \\ \hline \end{array}$$

we have $a^6 - b^6$,

which is the product required.

239. We shall perform this calculation in a still different manner, first multiplying the Ist. factor by the IVth. and next the II^{d}. by the III^{d}.

$$\begin{array}{l} \text{IV. } aa - ab + bb \\ \text{I. } a + b \\ \hline a^3 - aab + abb \\ \quad abb - abb + b^3 \\ \hline \text{I. IV.} = a^3 + b^3. \end{array}$$

$$\begin{array}{l} \text{II. } aa + ab + bb \\ \text{III. } a - b \\ \hline a^3 + aab + abb \\ \quad - aab - abb - b^3 \\ \hline \text{II. III.} = a^3 - b^3. \end{array}$$

It remains to multiply the product I. IV. and II. III.

$$\begin{array}{l} \text{I. IV.} = a^3 - b^3 \\ \text{II. III.} = a^3 - b^3 \\ \hline a^6 + a^3b^3 \\ \quad - a^3b^3 - b^6 \\ \hline \end{array}$$

and we still obtain $a^6 - b^6$, as before.

240. It will be proper to illustrate this example by a numerical application. Let us make $a = 3$ and $b = 2$, we shall have $a + b = 5$ and $a - b = 1$; further, $aa = 9$, $ab = 6$, $bb = 4$. Therefore $aa + ab + bb = 19$, and $aa - ab + bb = 7$. So that the product required is that of $5 \times 19 \times 1 \times 7$, which is 665.

Now $a^6 = 729$, and $b^6 = 64$, consequently the product required is $a^6 - b^6 = 665$, as we have already seen.

CHAPTER IV.

Of the Division of Compound Quantities.

241. When we wish simply to represent division, we make use of the usual mark of fractions, which is, to write the denominator under the numerator, separating them by a line; or to inclose each quantity between parentheses, placing two points between the divisor and dividend. If it were required, for example, to divide $a+b$ by $c+d$, we should represent the quotient thus $\frac{a+b}{c+d}$, according to the former method; and thus, $(a+b):(c+d)$ according to the latter. Each expression is read $a+b$ divided by $c+d$.

242. *When it is required to divide a compound quantity by a simple one, we divide each term separately.* For example:

$$6a-8b+4c, \text{ divided by } 2, \text{ gives } 3a-4b+2c;$$
$$\text{and } (aa-2ab):(a)=a-2b.$$

In the same manner

$$(a^3-2aab+3aab):(a)=aa-2ab+3ab;$$
$$(4aab-6aac+8abc):(2a)=2ab-3ac+4bc;$$
$$(9aabc-12abbc+15abcc):(3abc)=3a-4b+5c, \&c.$$

243. If it should happen that a term of the dividend is not divisible by the divisor, the quotient is represented by a fraction, as in the division of $a+b$ by a, which gives $1+\frac{b}{a}$. Likewise,

$$(aa-ab+bb):(aa)=1-\frac{b}{a}+\frac{bb}{aa}.$$

For the same reason, if we divide $2a+b$ by 2, we obtain $a+\frac{b}{2}$; and here it may be remarked, that we may write $\frac{1}{2}b$, instead of $\frac{b}{2}$, because $\frac{1}{2}$ times b is equal to $\frac{b}{2}$. In the same manner $\frac{b}{3}$ is the same as $\frac{1}{3}b$, and $\frac{2b}{3}$ the same as $\frac{2}{3}b$, &c.

244. But when the divisor is itself a compound quantity, division becomes more difficult. Sometimes it occurs where we least expect it; but when it cannot be performed, we must content ourselves

with representing the quotient by a fraction, in the manner that we have already described. Let us begin by considering some cases, in which actual division succeeds.

245. Suppose it were required to divide the dividend $ac - bc$ by the divisor $a - b$, *the quotient must* then *be such as, when multiplied by the divisor* $a - b$, *will produce the dividend* $ac - bc$. Now it is evident, that this quotient must include c, since without it we could not obtain ac. In order, therefore, to try whether c is the whole quotient, we have only to multiply it by the divisor, and see if that multiplication produces the whole dividend, or only part of it. In the present case, if we multiply $a - b$ by c, we have $ac - bc$, which is exactly the dividend; so that c is the whole quotient. It is no less evident, that

$$(aa + ab) : (a + b) = a;\ (3aa - 2ab) : (3a - 2b) = a;$$
$$(6aa - 9ab) : (2a - 3b) = 3a, \text{ \&c.}$$

246. ***We cannot fail, in this way, to find a part of the quotient; if, therefore, what we have found, when multiplied by the divisor, does not yet exhaust the dividend, we have only to divide the remainder again by the divisor, in order to obtain a second part of the quotient; and to continue the same method, until we have found the whole quotient.***

Let us, as an example, divide $aa + 3ab + 2bb$ by $a + b$; it is evident, in the first place, that the quotient will include the term a, since otherwise we should not obtain aa. Now, from the multiplication of the divisor $a + b$ by a, arises $aa + ab$; which quantity being subtracted from the dividend, leaves a remainder $2ab + 2bb$. This remainder must also be divided by $a + b$; and it is evident that the quotient of this division must contain the term $2b$. Now $2b$ multiplied by $a + b$, produces exactly $2ab + 2bb$; consequently $a + 2b$ is the quotient required; which, multiplied by the divisor $a + b$, ought to produce the dividend $aa + 3ab + 2bb$. See the whole operation:

$$\begin{array}{l} a + b)\ aa + 3ab + 2bb\ (a + 2b \\ \qquad\ \ \underline{aa + ab\qquad\qquad} \\ \qquad\qquad\quad 2ab + 2bb \\ \qquad\qquad\quad \underline{2ab + 2bb} \\ \qquad\qquad\qquad\ \ 0. \end{array}$$

247. *This operation will be facilitated if we choose one of the terms of the divisor to be written first, and then, in arranging the terms of the dividend, begin with the highest powers of that first term of the divisor.* This term in the preceding example was a; the following examples will render the operation more clear.

$$
\begin{array}{l}
a - b)\ a^3 - 3aab + 3abb - b^3\ (aa - 2ab + bb \\
\qquad a^3 - aab \\
\hline
\qquad -2aab + 3abb \\
\qquad -2aab + 2abb \\
\hline
\qquad\qquad abb - b^3 \\
\qquad\qquad abb - b^3 \\
\hline
\qquad\qquad 0.
\end{array}
$$

$$
\begin{array}{l}
a + b)\ aa - bb\ (a - b \\
\qquad aa + ab \\
\hline
\qquad -ab - bb \\
\qquad -ab - bb \\
\hline
\qquad 0.
\end{array}
$$

$$
\begin{array}{l}
3a - 2b)\ 18aa - 8bb\ (6a + 4b \\
\qquad 18aa - 12ab \\
\hline
\qquad\qquad 12ab - 8bb \\
\qquad\qquad 12ab - 8bb \\
\hline
\qquad\qquad 0.
\end{array}
$$

$$
\begin{array}{l}
a + b)\ a^3 + b^3\ (aa - ab + bb \\
\qquad a^3 + aab \\
\hline
\qquad -aab + b^3 \\
\qquad -aab - abb \\
\hline
\qquad\qquad abb + b^3 \\
\qquad\qquad abb + b^3 \\
\hline
\qquad\qquad 0.
\end{array}
$$

$2a - b)\, 8a^3 - b\; (4aa + 2ab + bb$

$8a - 4aab$

$4aab - b_3$

$4aab - 2abb$

$2abb - b^3$

$2abb - b^3$

0.

$aa - 2ab + bb)\, a^4 - 4a^3b + 6aabb - 4ab^3 + b^4$

$aa - 2ab + bb)\, a^4 - 2a^3b + aabb$

$-2a^3b + 5aabb - 4ab^3$

$-2a^3b + 4aabb - 2ab^3$

$aabb - 2ab^3 + b^4$

$aabb - 2ab^3 + b^4$

0.

$aa - 2ab + 4bb)\, a^4 + 4aabb + 16b^4\; (aa + 2ab + 4bb$

$a^4 - 2a^3b + 4aabb$

$2a^3b + 16b^4$

$2a^3b - 4aabb + 8ab^3$

$4aabb - 8ab^3 + 16b^4$

$4aabb - 8ab^3 + 16b^4$

0.

$aa - 2ab + 2bb)\, a^4 + 4b^4\; (aa + 2ab + 2bb$

$a^4 - 2a^3b + 2aabb$

$2a^3b - 2aabb + 4b^4$

$2a^3b - 4aabb + 4ab^3$

$2aabb - 4ab^3 + 4b^4$

$2aabb - 4ab^3 + 4b^4$

0.

$$
\begin{array}{r}
1 - 2x + xx)\ 1 - 5x + 10xx - 10x^3 + 5x^4 - x^5 \\
1 - 3x + 3xx - x^3)\ 1 - 2x + xx \qquad\qquad\qquad\qquad \\
\hline
-3x + 9xx - 10x^3 \qquad\qquad \\
-3x + 6xx - 3x^3 \qquad\qquad \\
\hline
3xx - 7x^3 + 5x^4 \qquad \\
3xx - 6x^3 + 3x^4 \qquad \\
\hline
-x^3 + 2x^4 - x^5 \\
-x^3 + 2x^4 - x^5 \\
\hline
0.
\end{array}
$$

CHAPTER V.

Of the Resolution of Fractions into Infinite Series.

248. When the dividend is not divisible by the divisor, the quotient is expressed, as we have already observed, by a fraction.

Thus, if we have to divide 1 by $1 - a$, we obtain the fraction $\frac{1}{1-a}$. This, however, does not prevent us from attempting the division, according to the rules that have been given, and continuing it as far as we please. We shall not fail to find the true quotient, though under different forms.

249. To prove this, let us actually divide the dividend 1 by the divisor $1 - a$, thus:

$$
\begin{array}{l}
1 - a)\ 1\ \left(1 + \frac{a}{1-a}\right. \\
\qquad\ \ 1 - a \\
\hline
\text{remainder} \quad a
\end{array}
\qquad ; \text{ or, } \qquad
\begin{array}{l}
1 - a)\ 1\ \left(1 + a + \frac{aa}{1-a}\right. \\
\qquad\ \ 1 - a \\
\hline
\qquad\ \ a \\
\qquad\ \ a - aa \\
\hline
\text{remainder } aa
\end{array}
$$

To find a greater number of forms, we have only to continue dividing aa by $1 - a$;

$$1-a)\,aa\,(aa+\frac{a^3}{1-a}, \text{ then } 1-a)\,a^3\,(a^3+\frac{a^4}{1-a}$$
$$\frac{aa-a^3}{a^3} \qquad\qquad \frac{a^3-a^4}{a^4}$$

and again $1-a)\,a^4\,(a^4+\dfrac{a^5}{1-a}$

$$\frac{a^4-a^5}{a^5}, \&c.$$

250. This shows that the fraction $\dfrac{1}{1-a}$ may be exhibited under all the following forms:

$$\text{I.)}\ 1+\frac{a}{1-a};\ \text{II.)}\ 1+a+\frac{aa}{1-a};$$
$$\text{III.)}\ 1+a+aa+\frac{a^3}{1-a};\ \text{IV.)}\ 1+a+aa+a^3+\frac{a^4}{1-a};$$
$$\text{V.)}\ 1+a+aa+a^3+a^4+\frac{a^5}{1-a}, \&c.$$

Now, by considering the first of these expressions, which is $1+\dfrac{a}{1-a}$, and remembering that 1 is the same as $\dfrac{1-a}{1-a}$, we have

$$1+\frac{a}{1-a}=\frac{1-a}{1-a}+\frac{a}{1-a}=\frac{1-a+a}{1-a}=\frac{1}{1-a}.$$

If we follow the same process with regard to the second expression $1+a+\dfrac{aa}{1-a}$, that is to say, if we reduce the integral part $1+a$ to the same denominator $1-a$, we shall have $\dfrac{1-aa}{1-a}$, to which if we add $+\dfrac{aa}{1-a}$, we shall have $\dfrac{1-aa+aa}{1-a}$, that is to say, $\dfrac{1}{1-a}$.

In the third expression $1+a+aa+\dfrac{a^3}{1-a}$, the integers reduced to the denominator $1-a$ make $\dfrac{1-a^3}{1-a}$; and if we add to that the fraction $\dfrac{a^3}{1-a}$, we have $\dfrac{1}{1-a}$; wherefore all these expressions are equal in value to $\dfrac{1}{1-a}$; the proposed fraction.

251. This being the case, we may continue the series as far as we please, without being under the necessity of performing any more calculations. We shall therefore have

$$\frac{1}{1-a} = 1 + a + aa + a^3 + a^4 + a^5 + a^6 + a^7 + \frac{a^8}{1-a};$$

or we might continue this further, and still go on without end. For this reason it may be said, that the proposed fraction has been resolved into an infinite series, which is

$$1+a+aa+a^3+a^4+a^5+a^6+a^7+a^8+a^9+a^{10}+a^{11}+a^{12}, \text{ \&c.}$$

to infinity. And there are sufficient grounds to maintain that the value of this infinite series is the same as that of the fraction $\frac{1}{1-a}$.

252. What we have said may, at first, appear surprising; but the consideration of some particular cases will make it easily understood.

Let us suppose, in the first place, $a = 1$; our series will become $1 + 1 + 1 + 1 + 1 + 1 + 1$, &c. The fraction $\frac{1}{1-a}$, to which it must be equal, becomes $\frac{1}{0}$. Now, we before remarked, that $\frac{1}{0}$ is a number infinitely great; which is, therefore, here confirmed in a satisfactory manner.

But if we suppose $a = 2$, our series becomes $= 1 + 2 + 4 + 8 + 16 + 32 + 64$, &c. to infinity, and its value must be $\frac{1}{1-2}$, that is to say, $\frac{1}{-1} = -1$; which at first sight will appear absurd. But it must be remarked, that if we wish to stop at any term of the above series, we cannot do so without joining the fraction which remains. Suppose, for example, we were to stop at 64, after having written $1 + 2 + 4 + 8 + 16 + 32 + 64$, we must join the fraction $\frac{128}{1-2}$, or $\frac{128}{-1}$, or -128; we shall therefore have $127 - 128$, that is in fact -1.

Were we to continue the series without intermission, the fraction indeed would be no longer considered, but then the series would still go on.

253. These are the considerations which are necessary, when we assume for a numbers greater than unity. But if we suppose a less than 1, the whole becomes more intelligible.

For example, let $a = \frac{1}{2}$; we shall have

$$\frac{1}{1-a} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2,$$

which will be equal to the following series:

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$, &c. to infinity.

Now, if we take only two terms of this series, we have $1 + \frac{1}{2}$, and it wants $\frac{1}{2}$, that it may be equal to $\frac{1}{1-a} = 2$. If we take three terms, it wants $\frac{1}{4}$; for the sum is $1\frac{3}{4}$. If we take four terms, we have $1\frac{7}{8}$, and the deficiency is only $\frac{1}{8}$. We see, therefore, that the more terms we take, the less the difference becomes, and that, consequently, if we continue on to infinity, there will be no difference at all between the sum of the series and 2, the value of the fraction $\frac{1}{1-a}$.

254. Let $a = \frac{1}{3}$; our fraction $\frac{1}{1-a}$ will be $= \frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1\frac{1}{2}$, which reduced to an infinite series, becomes

$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}\ \frac{1}{243}$, &c.

and to which $\frac{1}{1-a}$ is consequently equal.

When we take two terms, we have $1\frac{1}{3}$, and there wants $\frac{1}{6}$. If we take three terms, we have $1\frac{4}{9}$, and there will still be wanting $\frac{1}{18}$. Take four terms, we shall have $1\frac{13}{27}$, and the difference is $\frac{1}{54}$. Since the error, therefore, always becomes three times less, it must evidently vanish at last.

255. Suppose $a = \frac{2}{3}$; we shall have $\frac{1}{1-a} = \frac{1}{1-\frac{2}{3}} = 3$, and the series $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243}$, &c. to infinity. Taking first $1\frac{2}{3}$, the error is $1\frac{1}{3}$; taking three terms, which make $2\frac{1}{9}$, the error is $\frac{8}{9}$; taking four terms we have $2\frac{11}{27}$, and the error is $\frac{16}{27}$.

256. If $a = \frac{1}{4}$, the fraction is $\frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = 1\frac{1}{3}$; and the series becomes $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$, &c. The two first terms making $1 + \frac{1}{4}$, will give $\frac{1}{12}$ for the error; and taking one term more, we have $1\frac{5}{16}$, that is to say, only an error of $\frac{1}{48}$.

257. In the same manner, we may resolve the fraction $\frac{1}{1+a}$, into

an infinite series by actually dividing the numerator 1 by the denominator $1 + a$, as follows:

$$\begin{array}{l}
1 + a)\ 1\ (1 - a + aa - a^3 + a^4 \\
\quad\quad\ \ \underline{1 + a} \\
\quad\quad\quad - a \\
\quad\quad\quad \underline{- a - aa} \\
\quad\quad\quad\quad\quad\ aa \\
\quad\quad\quad\quad\quad\ \underline{aa + a^3} \\
\quad\quad\quad\quad\quad\quad\quad - a^3 \\
\quad\quad\quad\quad\quad\quad\quad \underline{- a^3 - a^4} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\ a^4 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \underline{a^4 + a^5} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad - a^5, \text{ \&c.}
\end{array}$$

Whence it follows that the fraction $\frac{1}{1 + a}$ is equal to the series

$$1 - a + aa - a^3 + a^4 - a^5 + a^6 - a^7, \text{ \&c.}$$

258. If we make $a = 1$, we have this remarkable comparison: $\frac{1}{1 + a} = \frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1$, &c. to infinity. This will appear rather contradictory; for if we stop at -1, the series gives 0; and if we finish by $+1$, it gives 1. But this is precisely what solves the difficulty; for since we must go on to infinity without stopping either at -1 or at $+1$, it is evident that the sum can neither be 0 nor 1, but that this result must lie between these two, and therefore be $= \frac{1}{2}$.

259. Let us now make $a = \frac{1}{2}$, and our fraction will be $\frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$, which must therefore express the value of the series

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \tfrac{1}{32} + \tfrac{1}{64}, \text{ \&c. to infinity.}$$

If we take only the two leading terms of this series, we have $\frac{1}{2}$, which is too small by $\frac{1}{6}$. If we take three terms, we have $\frac{3}{4}$, which is too much by $\frac{1}{12}$. If we take four terms, we have $\frac{5}{8}$, which is too small by $\frac{1}{24}$, &c.

260. Suppose again $a = \frac{1}{3}$; our fraction will be $= \frac{1}{1 + \frac{1}{3}} = \frac{3}{4}$,

and to this the series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \frac{1}{729}$, &c. continued to infinity, must be equal. Now, by considering only two terms, we have $\frac{2}{3}$, which is too small by $\frac{1}{12}$. Three terms make $\frac{7}{9}$, which is too much by $\frac{1}{36}$. Four terms make $\frac{20}{27}$, which is too small by $\frac{1}{108}$, and so on.

261. The fraction $\frac{1}{1 + a}$ may also be resolved into an infinite series another way; namely, by dividing 1 by $a + 1$, as follows:

$$a + 1)\ 1\ (\frac{1}{a} - \frac{1}{aa} + \frac{1}{a^3} - \frac{1}{a^4} + \frac{1}{a^5}$$

$$\begin{array}{l}
1 + \frac{1}{a} \\
\hline
-\frac{1}{a} \\
-\frac{1}{a} - \frac{1}{aa} \\
\hline
\frac{1}{aa} \\
\frac{1}{aa} + \frac{1}{a^3} \\
\hline
-\frac{1}{a^3} \\
-\frac{1}{a^3} - \frac{1}{a^4} \\
\hline
\frac{1}{a^4} \\
\frac{1}{a^4} + \frac{1}{a^5} \\
\hline
-\frac{1}{a^5}, \text{ \&c.}
\end{array}$$

Consequently, our fraction $\frac{1}{a + 1}$, is equal to the infinite series $\frac{1}{a} - \frac{1}{aa} + \frac{1}{a^3} - \frac{1}{a^4} + \frac{1}{a^5} - \frac{1}{a^6}$, &c. Let us make $a = 1$, and we shall have the series

$$-1 + 1 - 1 + 1 + 1, \text{ \&c.} = \tfrac{1}{2}, \text{ as before.}$$

And if we suppose $a = 2$, we shall have the series

$$\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{1}{16} + \tfrac{1}{32} - \tfrac{1}{64} \text{ \&c.} = \tfrac{1}{3}.$$

262. In the same manner, by resolving the general fraction $\frac{c}{a+b}$ into an infinite series, we shall have,

$$a+b)\ c\ (\frac{c}{a} - \frac{bc}{aa} + \frac{bbc}{a^3} - \frac{b^3c}{a^4}$$

$$c + \frac{bc}{a}$$

$$-\frac{bc}{a}$$

$$-\frac{bc}{a} - \frac{bbc}{aa}$$

$$\frac{bbc}{aa}$$

$$\frac{bbc}{aa} + \frac{b^3c}{a^3}$$

$$-\frac{b^3c}{a^3}$$

$$-\frac{b^3c}{a^3} - \frac{b^4c}{a^4}$$

$$\frac{b^4c}{a^4};$$

Whence it appears, that we may compare $\frac{c}{a+b}$ with the series

$$\frac{c}{a} - \frac{bc}{aa} + \frac{bbc}{a^3} - \frac{b^3c}{a^4}, \text{ \&c. to infinity.}$$

Let $a = 2$, $b = 4$, $c = 3$, and we shall have

$$\frac{c}{a+b} = \frac{3}{2+4} = \tfrac{3}{6} = \tfrac{1}{2} = \tfrac{3}{2} - 3 + 6 - 12, \text{ \&c.}$$

Let $a = 10$, $b = 1$, and $c = 11$, and we have

$$\frac{c}{a+b} = \frac{11}{10-1} = 1 = \tfrac{11}{10} - \tfrac{11}{100} + \tfrac{11}{1000} - \tfrac{11}{10000}, \text{ \&c.}$$

If we consider only one term of this series, we have $\frac{11}{10}$, which is too much by $\frac{1}{10}$; if we take two terms, we have $\frac{99}{100}$, which is too small by $\frac{1}{100}$; if we take three terms, we have $\frac{1001}{1000}$, which is too much by $\frac{1}{1000}$, &c.

263. When there are more than two terms in the divisor, we may also continue the division to infinity in the same manner.

Thus, if the fraction $\frac{1}{1-a+aa}$ were proposed, the infinite series to which it is equal would be found as follows:

$$
\begin{array}{l}
1-a+aa)\,1 \qquad (1+a-a^3-a^4+a^6,+a^7,\ \&c.\\
\quad 1-a+aa\\
\hline
\quad a-aa\\
\quad a-aa+a^3\\
\hline
\qquad -a^3\\
\qquad -a^3+a^4-a^5\\
\hline
\qquad\quad -a^4+a^5\\
\qquad\quad -a^4+a^5-a^6\\
\hline
\qquad\qquad a^6\\
\qquad\qquad a^6-a^7+a^8\\
\hline
\qquad\qquad\quad a^7-a^8\\
\qquad\qquad\quad a^6-a^8+a^7\\
\hline
\qquad\qquad\qquad -a^9
\end{array}
$$

We have therefore the equation of

$$\frac{1}{1-a+aa}=1+a-a^3-a^4+a^6+a^7-a^9-a^{10},\ \&c.$$

Here, if we make $a=1$, we have

$$1=1+1-1-1+1+1-1-1+1+1,\ \&c.$$

which series contains twice the series found above,

$$1-1+1-1+1,\ \&c.$$

Now, as we have found this $=\frac{1}{2}$, it is not astonishing that we should find $\frac{2}{2}$, or 1, for the value of that which we have just determined.

Make $a=\frac{1}{2}$, and we shall then have the equation

$$\frac{1}{\frac{3}{4}}=\frac{4}{3}=1+\frac{1}{2}-\frac{1}{8}-\frac{1}{16}+\frac{1}{64}+\frac{1}{128}-\frac{1}{512},\ \&c.$$

Suppose $a=\frac{1}{3}$, we shall have the equation

$$\frac{1}{\frac{7}{9}}=\frac{9}{7}=1+\frac{1}{3}-\frac{1}{27}-\frac{1}{81}+\frac{1}{729},\ \&c.$$

If we take the four leading terms of this series, we have $\frac{104}{81}$, which is only $\frac{1}{567}$, less than $\frac{9}{7}$.

Suppose again $a=\frac{2}{3}$, we shall have

$$\frac{1}{\frac{7}{9}} = \frac{9}{7} = 1 + \frac{2}{3} - \frac{8}{27} - \frac{16}{81} + \frac{64}{729}, \text{ \&c.}$$

This series must therefore be equal to the preceding one; and subtracting one from the other, $\frac{1}{3} - \frac{7}{27} - \frac{16}{81} + \frac{63}{729}$, must be $= 0$. These four terms added together make $- \frac{2}{81}$.

264. The method which we have explained, serves to resolve, generally, all fractions into infinite series; and, therefore, it is often found to be of the greatest utility. Further, it is remarkable, that *an infinite series, though it never ceases, may have a determinate value.* It may be added, that from this branch of mathematics inventions of the utmost importance have been derived, on which account the subject deserves to be studied with the greatest attention.

CHAPTER VI.

Of the Squares of Compound Quantities.

265. When it is required to find the square of a compound quantity, we have only to multiply it by itself, and the product will be the square required.

For example, the square of $a + b$ is found in the following manner:

$$\begin{array}{l} a + b \\ a + b \\ \hline a a + a b \\ \quad\;\; a b + b b \\ \hline a a + 2 a b + b b. \end{array}$$

266. So that, *when the root consists of two terms added together,* as $a + b$, *the square comprehends,* 1*st, the square of each term,* namely, $a a$, and $b b$; 2*dly, twice the product of the two terms,* namely, $2 a b$. So that the sum $a a + 2 a b + b b$ is the square of $a + b$. Let, for example, $a = 10$ and $b = 3$, that is to say, let it be required to find the square of 13, we shall have $100 + 60 + 9$, or 169.

267. We may easily find, by means of this formula, the squares of numbers, however great, if we divide them into two parts. To find, for example, the square of 57, we consider that this number is $= 50 + 7$; whence we conclude that its square is

$$= 2500 + 700 + 49 = 3249.$$

268. Hence it is evident that the square of $a + 1$ will be $aa + 2a + 1$; now since the square of a is $a\,a$, we find the square $a + 1$ by adding to that $2a + 1$; and it must be observed, that this $2a + 1$ is the sum of the two roots a and $a + 1$.

Thus, as the square of 10 is 100, that of 11 will be $100 + 21$. The square of 57 being 3249, that of 58 is $3249 + 115 = 3364$. The square of $59 = 3364 + 117 = 3481$; the square of

$$60 = 3481 + 119 = 3600, \text{ \&c.}$$

269. The square of a compound qüantity, as $a + b$, is represented in this manner: $(a + b)^2$. We have then

$$(a + b)^2 = a\,a + 2\,a\,b + b\,b,$$

whence we deduce the following equations:

$$(a + 1)^2 = a\,a - 2\,a + 1;\ (a + 2)^2 = a\,a + 4\,a + 4;$$
$$(a + 3)^2 = a\,a + 6\,a + 9;\ (a + 4)^2 = aa + 8\,a + 16; \text{ \&c.}$$

270. *If the root is* a — b, *the square of it is* a a — 2 a b + b b, *which contains also the squares of the two terms, but in such a manner that we must take from their sum twice the product of those two terms.*

Let, for example, $a = -10$ and $b = -1$, the square of 9 will be found $= 100 - 20 + 1 = 81$.

271. Since we have the equation $(a - b)^2 = a\,a - 2\,a\,b + b\,b$, we shall have $(a - 1)^2 = a\,a - 2\,a + 1$. *The square of* a — 1 *is found,* therefore, *by subtracting from* a a *the sum of the two roots* a *and* a — 1, *namely,* 2 a — 1. Let, for example, $a = 50$, we have $a\,a = 2500$, and $a - 1 = 49$; then $49^2 = 2500 - 99 = 2401$.

272. What we have said may be also confirmed and illustrated by fractions. For if we take as the root $\frac{3}{5} + \frac{2}{5}$ (which make 1) the squares will be:

$$\tfrac{9}{25} + \tfrac{4}{25} + \tfrac{12}{25} = \tfrac{25}{25}, \text{ that is } 1.$$

Further, the square of $\frac{1}{2} - \frac{1}{3}$ (or of $\frac{1}{6}$) will be

$$\tfrac{1}{4} - \tfrac{1}{3} + \tfrac{1}{9} = \tfrac{1}{36}.$$

273. When the root consists of a greater number of terms, the method of determining the square is the same. Let us find, for example, *the square of* a + b + c.

$$\begin{array}{l} a+b+c \\ a+b+c \\ \hline aa+ab+ac \qquad +bc \\ \qquad ab+ac+bb+bc+cc \\ \hline aa+2ab+2ac+bb+2bc+cc. \end{array}$$

We see that it *includes, first, the square of each term of the root, and beside that, the double products of those terms multiplied two by two.*

274. To illustrate this by an example, let us divide the number 256 into three parts, 200 + 50 + 6; its square will then be composed of the following parts:

$$\begin{array}{r} 40000 \\ 2500 \\ 36 \\ 20000 \\ 2400 \\ 600 \\ \hline 65536 \end{array} \qquad\qquad \begin{array}{r} 256 \\ 256 \\ \hline 1536 \\ 1280 \\ 512 \\ \hline 65536 \end{array}$$

which is evidently equal to the product of 256 × 256.

275. *When some terms of the root are negative, the square is still found by the same rule; but we must take care what signs we prefix to the double products.* Thus, the square of $a - b - c$ being $aa + bb + cc - 2ab - 2ac + 2bc$, if we represent the number 256 by 300 — 40 — 4, we shall have,

$$\begin{array}{r} \text{Positive Parts.} \\ +90000 \\ 1600 \\ 320 \\ 16 \\ \hline +91936 \\ -26400 \\ \hline \end{array} \qquad\qquad \begin{array}{r} \text{Negative Parts.} \\ -24000 \\ -2400 \\ \hline -26400 \end{array}$$

65536, the square of 256, as before.

CHAPTER VII.

Of the Extraction of Roots applied to Compound Quantities.

276. In order to give a certain rule for this operation, we must consider attentively the square of the root $a + b$, which is

$$a\,a + 2\,a\,b + b\,b,$$

that we may reciprocally find the root of a given square.

277. We must consider therefore, first, that as the square $a\,a + 2\,a\,b + b\,b$ is composed of several terms, it is certain that the root also will comprise more than one term; and that if we write the square in such a manner that the powers of one of the letters, as a, may go on continually diminishing, the first term will be the square of the first term of the root. And since, in the present case, the first term of the square is $a\,a$, it is certain that the first term of the root is a.

278. Having, therefore, found the first term of the root, that is to say a, we must consider the rest of the square, namely, $2\,a\,b + b\,b$, to see if we can derive from it the second part of the root, which is b. Now this remainder $2\,a\,b + b\,b$ may be represented by the product, $(2\,a + b)\,b$. Wherefore the remainder having two factors, $2\,a + b$, and b, it is evident that we shall find the latter, b, which is the second part of the root, by dividing the remainder $2\,a\,b + b\,b$ by $2\,a + b$.

279. So that the quotient, arising from the division of the above remainder by $2\,a + b$ is the second term of the root required. Now in this division we observe, that $2\,a$ is the double of the first term a, which is already determined. So that although the second term is yet unknown, and it is necessary, for the present, to leave its place empty, we may nevertheless attempt the division, since in it we attend only to the first term $2\,a$. But as soon as the quotient is found, which is here b, we must put it in the empty place, and thus render the division complete.

280. The calculation, therefore, by which we find the root of the square $a\,a + 2\,a\,b + b\,b$, may be represented thus:

$$\begin{array}{rl} & aa + 2ab + bb\ (a + b \\ & aa \\ \hline 2a + b) & 2ab + bb \\ & 2ab + bb \\ \hline & 0. \end{array}$$

281. We may, in the same manner, find the square root of other compound quantities, provided they are squares, as the following examples will show.

$$\begin{array}{rl} & aa + 6ab + 9bb\ (a + 3b \\ & aa \\ \hline 2a + 3b) & 6ab + 9bb \\ & 6ab + 9bb \\ \hline & 0. \end{array}$$

$$\begin{array}{rl} & 4aa - 4ab + bb\ (2a - b \\ & 4aa \\ \hline 4a - b) & -4ab + bb \\ & -4ab + bb \\ \hline & 0. \end{array}$$

$$\begin{array}{rl} & 9pp + 24pq + 16qq\ (3p + 4q \\ & 9pp \\ \hline 6p + 4q) & 24pq + 16qq \\ & 24pq + 16qq \\ \hline & 0. \end{array}$$

$$\begin{array}{rl} & 25xx - 60x + 36\ (5x - 6 \\ & 25xx \\ \hline 10x - 6) & -60x + 36 \\ & -60x + 36 \\ \hline & 0. \end{array}$$

282. When there is a remainder after the division, it is a proof that the root is composed of more than two terms. We then consider the two terms already found as forming the first part, and endeavour to derive the other from the remainder, in the same manner as we found the second term of the root. The following example will render this operation more clear.

$$aa + 2ab - 2ac - 2bc + bb + cc \,(a + b - c$$
$$aa$$
$$2a + b)\; 2ab - 2ac - 2bc + bb + cc$$
$$2ab \qquad\qquad + bb$$
$$2a + 2b - c)\; -2ac - 2bc + cc$$
$$-2ac - 2bc + cc$$
$$0.$$

$$a^4 + 2a^3 + 3aa + 2a + 1 \,(aa + a + 1$$
$$a^4$$
$$2aa + a)\; 2a^3 + 3aa$$
$$2a^3 + \; aa$$
$$2aa + 2a + 1)\; 2aa + 2a + 1$$
$$2aa + 2a + 1$$
$$0.$$

$$a^4 - 4a^3b + 8ab^3 + 4b^4 \,(aa - 2ab - 2bb$$
$$a^4$$
$$2aa - 2ab)\; -4a^3b + 8ab^3 + 4b^4$$
$$-4a^3b + 4aabb$$
$$2aa - 4ab) - 2bb)\; -4aabb + 8ab^3 + 4b^4$$
$$-4aabb + 8ab^3 + 4b^4$$
$$0.$$

$$\begin{array}{l}
a^6 - 6a^5b + 15a^4bb - 20a^3b^3 + 15aab^4 - 6ab^5 + b^6 \\
a^6 \qquad\qquad\qquad\qquad\qquad (a^3 - 3aab + 3abb - b^3 \\
\hline
2a^3 - 3aab)\; - 6a^5b + 15a^4bb \\
\qquad\qquad\;\; - 6a^5b + 9a^4bb \\
\hline
2a^3 - 6aab + 3abb)\; 6a^4bb - 20a^3b^3 + 15aab^4 \\
\qquad\qquad\qquad\qquad\;\; 6a^4bb - 18a^3b^3 + 9aab^4 \\
\hline
2a^3 - 6aab + 6abb - b^3)\; - 2a^3b^3 + 6aab^4 - 6ab^5 + b^6 \\
\qquad\qquad\qquad\qquad\qquad\;\; - 2a^3b^3 + 6aab^4 - 6ab^5 + b^6 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0.
\end{array}$$

283. We easily deduce from the rule which we have explained, the method which is taught in books of arithmetic for the extraction of the square root. Some examples in numbers:

$$\begin{array}{rl}
 & \dot{5}2\dot{9}\ (23 \\
 & 4 \\
\hline
43) & 129 \\
 & 129 \\
\hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & \dot{1}7\dot{6}4\ (42 \\
 & 16 \\
\hline
82) & 164 \\
 & 164 \\
\hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & \dot{2}3\dot{0}4\ (48 \\
 & 16 \\
\hline
88) & 704 \\
 & 704 \\
\hline
 & 0.
\end{array}$$

$$\begin{array}{rl}
 & \dot{4}0\dot{9}6\ (64 \\
 & 36 \\
\hline
124) & 496 \\
 & 496 \\
\hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & \dot{9}6\dot{0}4\ (98 \\
 & 81 \\
\hline
188) & 1504 \\
 & 1504 \\
\hline
 & 0.
\end{array}$$

$$\begin{array}{rl}
 & \dot{1}5\dot{6}2\dot{5}\ (125 \\
 & 1 \\
\hline
22) & 56 \\
 & 44 \\
\hline
245) & 1225 \\
 & 1225 \\
\hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & \dot{9}9\dot{8}0\dot{0}1\ (999 \\
 & 81 \\
\hline
189) & 1880 \\
 & 1701 \\
\hline
1989) & 17901 \\
 & 17901 \\
\hline
 & 0.
\end{array}$$

284. But when there is a remainder after the whole operation, it is a proof that the number proposed is not a square, and consequently that its root cannot be assigned. In such cases, the radical sign, which we before employed, is made use of. It is written before the quantity, and the quantity itself is placed between parentheses, or under a line. Thus, the square root of $a\,a + b\,b$ is represented by $\sqrt{(aa+bb)}$, or by $\sqrt{aa+bb}$; and $\sqrt{(1-xx)}$, or $\sqrt{1-xx}$, expresses the square root of $1 - x\,x$. Instead of this radical sign, we may use the fractional exponent $\frac{1}{2}$, and represent the square root of $a\,a + b\,b$, for instance, by $(a\,a + b\,b)^{\frac{1}{2}}$, or by $\overline{aa+bb}\rceil^{\frac{1}{2}}$.

CHAPTER VIII.

Of the Calculation of Irrational Quantities.

285. When it is required to add together two or more irrational quantities, this is done, according to the method before laid down, by writing all the terms in succession, each with its proper sign. And with regard to abbreviation, we must remark that *instead of* $\sqrt{a} + \sqrt{a}$, *for example, we write* $2\sqrt{a}$; *and* that $\sqrt{a} - \sqrt{a} = 0$, because these two terms destroy one one another. Thus, *the quantities* $3 + \sqrt{2}$ *and* $1 + \sqrt{2}$, *added together, make* $4 + 2\sqrt{2}$ *or* $4 + \sqrt{8}$; the sum of $5 + \sqrt{3}$ and $4 - \sqrt{3}$ is 9; and that of $2\sqrt{3} + 3\sqrt{2}$ and $\sqrt{3} - \sqrt{2}$ is $3\sqrt{3} + 2\sqrt{2}$.

286. Subtraction also is very easy, since we have only to add the proposed numbers, changing first their signs; the following example will show this; let us subtract the lower number from the upper.

$$\begin{array}{l} 4 - \sqrt{2} + 2\sqrt{3} - 3\sqrt{5} + 4\sqrt{6} \\ 1 + 2\sqrt{2} - 2\sqrt{3} - 5\sqrt{5} + 6\sqrt{6} \\ \hline 3 - 3\sqrt{2} + 4\sqrt{3} + 2\sqrt{5} - 2\sqrt{6} \end{array}$$

287. In multiplication we must recollect that $\sqrt{a}$ *multiplied by* $\sqrt{a}$ *produces* a; and that *if the numbers which follow the sign* $\sqrt{}$ *are different, as* a *and* b, *we have* $\sqrt{ab}$ *for the product of* $\sqrt{a}$ *multiplied by* $\sqrt{b}$. After this it will be easy to perform the following examples:

$$\begin{array}{l} 1 + \sqrt{2} \\ 1 + \sqrt{2} \\ \hline 1 + \sqrt{2} \\ \quad + \sqrt{2} + 2 \\ \hline 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2} \end{array} \qquad \begin{array}{l} 4 + 2\sqrt{2} \\ 2 - \quad \sqrt{2} \\ \hline 8 + 4\sqrt{2} \\ \quad - 4\sqrt{2} - 4 \\ \hline 8 - 4 = 4 \end{array}$$

288. What we have said applies also to imaginary quantities; we shall only observe further, that $\sqrt{-a}$ *multiplied by* $\sqrt{-a}$ *produces* $-a$.

If it were required to find the cube of $-1 + \sqrt{-3}$, we should take the square of that number, and then multiply that square by the same number; see the operation:

$$\begin{array}{l} -1 + \sqrt{-3} \\ -1 + \sqrt{-3} \\ \hline \quad 1 - \sqrt{-3} \\ \quad\quad - \sqrt{-3} - 3 \\ \hline 1 - 2\sqrt{-3} - 3 = -2 - 2\sqrt{-3} \\ \qquad\qquad\qquad\quad -1 + \quad \sqrt{-3} \\ \hline \qquad\qquad\qquad\qquad 2 + 2\sqrt{-3} \\ \qquad\qquad\qquad\qquad\quad - 2\sqrt{-3} + 6 \\ \hline \qquad\qquad\qquad\qquad 2 + 6 = 8. \end{array}$$

289. *In the division of surds, we have only to express the proposed quantities in the form of a fraction; this may be then changed into another expression having a rational denominator.* For if the denominator be $a + \sqrt{b}$, for example, and we multiply both it and the numerator by $a - \sqrt{b}$, the new denominator will be $a\,a - b$, in which there is no radical sign. Let it be proposed to divide $3 + 2\sqrt{2}$ by $1 + \sqrt{2}$; we shall first have $\frac{3 + 2\sqrt{2}}{1 + \sqrt{2}}$. Multiplying now the two terms of the fraction by $1 - \sqrt{2}$, we shall have for the numerator:

$$\begin{array}{l} 3+2\sqrt{2} \\ 1-\ \ \sqrt{2} \\ \hline 3+2\sqrt{2} \\ \quad -3\sqrt{2}-4 \\ \hline 3-\ \ \sqrt{2}-4=-\sqrt{2}-1; \end{array}$$

and for the denominator:

$$\begin{array}{l} 1+\sqrt{2} \\ 1-\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad -\sqrt{2}-2 \\ \hline 1-2=-1 \end{array}$$

Our new fraction therefore is $\frac{-\sqrt{2}-1}{-1}$; and if we again multiply the terms by -1, we shall have for the numerator $\sqrt{2}+1$, and for the denominator $+1$. Now it is easy to show that $\sqrt{2}+1$ is equal to the proposed fraction $\frac{3+2\sqrt{2}}{1+\sqrt{2}}$; for $\sqrt{2}+1$ being multiplied by the divisor $1+\sqrt{2}$, thus,

$$\begin{array}{l} 1+\sqrt{2} \\ 1+\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad +\sqrt{2}+2 \\ \hline \end{array}$$

we have $1+2\sqrt{2}+2=3+2\sqrt{2}$.

Another example: $8-5\sqrt{2}$ divided by $3-2\sqrt{2}$ makes $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$. Multiplying the two terms of this fraction by $3+2\sqrt{2}$, we have for the numerator,

$$\begin{array}{l} 8-5\sqrt{2} \\ 3+2\sqrt{2} \\ \hline 24-15\sqrt{2} \\ \quad +16\sqrt{2}-20 \\ \hline 24+\sqrt{2}-20=4+\sqrt{2}; \end{array}$$

and for the denominator,

$$\begin{array}{l} 3 - 2\sqrt{2} \\ 3 + 2\sqrt{2} \\ \hline 9 - 6\sqrt{2} \\ \quad + 6\sqrt{2} - 8 \\ \hline 9 - 8 = +1. \end{array}$$

Consequently the quotient will be $4 + \sqrt{2}$. The truth of this may be proved in the following manner:

$$\begin{array}{l} 4 + \sqrt{2} \\ 3 - 2\sqrt{2} \\ \hline 12 + 3\sqrt{2} \\ \quad - 8\sqrt{2} - 4 \\ \hline 12 - 5\sqrt{2} - 4 = 8 - 5\sqrt{2}. \end{array}$$

290. In the same manner, we may transform such fractions into others, that have rational denominators. If we have, for example, the fraction $\frac{1}{5 - 2\sqrt{6}}$, and multiply its numerator and denominator by $5 + 2\sqrt{6}$, we transform it into this

$$\frac{5 + 2\sqrt{6}}{1} = 5 + 2\sqrt{6}.$$

In like manner the fraction $\frac{2}{-1 + \sqrt{-3}}$ assumes this form,

$$\frac{2 + 2\sqrt{-3}}{-4} = \frac{1 + \sqrt{-3}}{-2}.$$

And $\frac{\sqrt{6} + \sqrt{5}}{\sqrt{6} - \sqrt{5}}$ becomes $= \frac{11 + 2\sqrt{30}}{1} = 11 + 2\sqrt{30}$.

291. *When the denominator contains several terms, we may in the same manner make the radical signs in it vanish one by one.* Let the fraction $\frac{1}{\sqrt{10} - \sqrt{2} - \sqrt{3}}$ be proposed; we first multiply these terms by $\sqrt{10} + \sqrt{2} + \sqrt{3}$, and obtain the fraction

$$\frac{\sqrt{10} + \sqrt{2} + \sqrt{3}}{5 - 2\sqrt{6}}.$$

Then multiplying its numerator and denominator by $5 + 2\sqrt{6}$, we have $5\sqrt{10} + 11\sqrt{2} + 9\sqrt{3} + 2\sqrt{60}$.

CHAPTER IX.

Of Cubes, and the Extraction of Cube Roots.

292. To find *the cube of a root* a + b, we only multiply its square $a a + 2 a b + b b$ again by $a + b$, thus,

$$\begin{array}{l} a a + 2 a b + b b \\ \quad a + b \\ \hline a^3 + 2 a a b + a b b \\ \qquad\quad a a b + 2 a b b + b^3 \\ \hline \end{array}$$

and the cube will be $= a^3 + 3 a a b + 3 a b b + b^3$.

It *contains*, therefore, *the cubes of the two parts of the root, and* beside that, $3 a a b + 3 a b b$, a quantity equal to $(3 a b) \times (a + b)$; that is, *the triple product of the two parts*, a *and* b, *multiplied by their sum.*

293. So that whenever a root is composed of two terms, it is easy to find its cube by this rule. For example, the number $5 = 3 + 2$; its cube is therefore $27 + 8 + 18 \times 5 = 125$.

Let $7 + 3 = 10$ be the root; the cube will be

$$343 + 27 + 63 \times 10 = 1000.$$

To find the cube of 36, let us suppose the root $36 = 30 + 6$, and we have for the power required,

$$27000 + 216 + 540 \times 36 = 46656.$$

294. But if, on the other hand, the cube be given, namely, $a^3 + 3 a a b, + 3 a b b + b^3$, and it be required to find its root, we must premise the following remarks:

First, when the cube is arranged according to the powers of one letter, we easily know by the first term a^3, the first term a of the root, since the cube of it is a^3; if, therefore, we subtract that cube from the cube proposed, we obtain the remainder,

$$3 a a b + 3 a b b + b^3,$$

which must furnish the second term of the root.

295. But as we already know that the second term is $+ b$, we have principally to discover how it may be derived from the above

remainder. Now that remainder may be expressed by two factors, as $(3\,a\,a + 3\,a\,b + b\,b) \times (b)$; if, therefore, we divide by $3\,a\,a + 3\,a\,b + b\,b$, we obtain the second part of the root $+\,b$, which is required.

296. But as this second term is supposed to be unknown, the divisor also is unknown; nevertheless we have the first term of that divisor, which is sufficient; for, it is $3\,a\,a$, that is, thrice the square of the first term already found; and by means of this, it is not difficult to find also the other part, b, and then to complete the divisor before we perform the division. For this purpose, it will be necessary to join to $3\,a\,a$ thrice the product of the two terms, or $3\,a\,b$, and $b\,b$, or the square of the second term of the root.

297. Let us apply what we have said to two examples of other given cubes.

I.
$$
\begin{array}{rl}
 & a^3 + 12\,a\,a + 48\,a + 64\ (a + 4 \\
 & a^3 \\
\hline
3\,a\,a + 12\,a + 16) & 12\,a\,a + 48\,a + 64 \\
 & 12\,a\,a + 48\,a + 64 \\
\hline
 & 0.
\end{array}
$$

II.
$$
\begin{array}{rl}
 & a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1 \\
 & a^6 \qquad\qquad\qquad\qquad (aa - 2a + 1 \\
\hline
3a^4 - 6a^3 + 4aa) & -6a^5 + 15a^4 - 20a^3 \\
 & -6a^5 + 12a^4 - 8a^3 \\
\hline
3a^4 - 12a^3 + 12aa + 3a^2 - 6a + 1) & 3a^4 - 12a^3 + 15aa - 6a + 1 \\
 & 3a^4 - 12a^3 + 15aa - 6a + 1 \\
\hline
 & 0.
\end{array}
$$

298. The analysis which we have given is the foundation of the common rule for the extraction of the cube root in numbers. An example of the operation in the number 2197:

```
         2197 (10 + 3 = 13
         1000
     ----------
  300|1197
   90|
    9|
 ----|
  399|1197
     ------
       0.
```

Let us also extract the cube root of 34965783 :

```
         34965783 (300 + 20 + 7
         27000000
        ---------
 270000| 7965783
  18000|
    400|
 ------|
 288400| 5768000
 ------|--------
 307200| 2197783
   6720|
     49|
 ------|--------
 313969| 2197783
 ------|--------
            0.
```

CHAPTER X.

Of the Higher Powers of Compound Quantities.

299. After squares and cubes come higher powers, or powers of a greater number of degrees. They are represented by exponents in the manner which we before explained: we have only to remember, when the root is compound, to inclose it in a parenthesis. Thus $(a + b)^5$ means that $a + b$ is raised to the fifth degree, and $(a - b)^6$ represents the sixth power of $a - b$. We shall in this chapter explain the nature of these powers.

300. Let $a + b$ be the root, or the first power, and the higher powers will be found by multiplication in the following manner:

$$
\begin{array}{rl}
(a+b)^1 = & a + b \\
& a + b \\
\hline
& a^2 + ab \\
& \quad + ab + bb \\
\hline
(a+b)^2 = & a^2 + 2ab + bb \\
& a + b \\
\hline
& a^3 + 2aab + abb \\
& \quad + aab + 2abb + b^3 \\
\hline
(a+b)^3 = & a^3 + 3aab + 3abb + b^3 \\
& a + b \\
\hline
& a^4 + 3a^3b + 3aabb + ab^3 \\
& \quad + a^3b + 3aabb + 3ab^3 + b^4 \\
\hline
(a+b)^4 = & a^4 + 4a^3b + 6aabb + 4ab^3 + b^4 \\
& a + b \\
\hline
& a^5 + 4a^4b + 6a^3bb + 4aab^3 + ab^4 \\
& \quad + a^4b + 4a^3bb + 6aab^3 + 4ab^4 + b^5 \\
\hline
(a+b)^5 = & a^5 + 5a^4b + 10a^3bb + 10aab^3 + 5ab^4 + b^5 \\
& a + b \\
\hline
& a^6 + 5a^5b + 10a^4bb + 10a^3b^3 + 5aab^4 + ab^5 \\
& \quad + a^5b + 5a^4bb + 10a^3b^3 + 10aab^4 + 5ab^5 + b^6 \\
\hline
(a+)^6 = & a^6 + 6a^5b + 15a^4bb + 20a^3b^3 + 15aab^4 + 6ab^5 + b^6
\end{array}
$$

301. The powers of the root $a - b$ are found in the same manner, and we shall immediately perceive that they do not differ from the preceding, excepting that the 2d, 4th, 6th, &c. terms are affected by the sign *minus*;

$$(a-b)^1 = a - b$$
$$a - b$$
$$aa - b$$
$$- ab + bb$$
$$(a-b)^2 = a^2 - 2ab + bb$$
$$a - b$$
$$a^3 - 2aab + abb$$
$$- aab + 2abb - b^3$$
$$(a-b)^3 = a^3 - 3aab + 3abb - b^3$$
$$a - b$$
$$a^4 - 3a^3b + 3aabb - ab^3$$
$$- a^3b + 3aabb - 3ab^3 + b^4$$
$$(a-b)^4 = a^4 - 4a^3b + 6aabb - 4ab^3 + b^4$$
$$a - b$$
$$a^5 - 4a^4b + 6a^3bb - 4aab^3 + ab^4$$
$$- a^4b + 4a^3bb - 6aab^3 + 4ab^4 - b^5$$
$$(a-b)^5 = a^5 - 5a^4b + 10a^3bb - 10aab^3 + 5ab^4 - b^5$$
$$a - b$$
$$a^6 - 5a^5b + 10a^4bb - 10a^3b^3 + 5aab^4 - ab^5$$
$$- a^5b + 5a^4bb + 10a^3b^3 + 10aab^4 - 5ab^5 + b^6$$
$$(a-b)^6 = a^6 - 6a^5b + 15a^4bb - 20a^3b^3 + 15aab^4 - 6ab^5 + b^6$$

Here we see that all the odd powers of b have the sign —, while the even powers retain the sign +. The reason of this is evident; for since $-b$ is the term of the root, the powers of that letter will ascend in the following series, $-b$, $+bb$, $-b^3$, $+b^4$, $-b^5$, $+a^6$, &c. which clearly shows that the even powers must be affected by the sign +, and the odd ones by the contrary sign —.

302. An important question occurs in this place; namely, how we may find, without being obliged always to perform the same calculation, all the powers either of $a + b$, or $a - b$.

We must remark, in the first place, that if we can assign all the powers of $a + b$, those of $a - b$ are also found, since we have only to change the signs of the even terms, that is to say, of the second, the fourth, the sixth, &c. The business then is to establish a rule, by which *any power of* a + b, *however high, may be determined without the necessity of calculating all the preceding ones.*

303. Now, if from the powers which we have already determined we take away the numbers that precede each term, which are called the *coefficients*, we observe in all the terms a singular order; *first, we see the first term* a *of the root raised to the power which is required; in the following terms the powers of* a *diminish continually by unity, and the powers of* b *increase in the same proportion; so that the sum of the exponents of* a *and of* b *is always the same, and always equal to the exponent of the power required; and, lastly, we find the term* b *by itself raised to the same power.* If, therefore, the tenth power of $a + b$ were required, we are certain that the terms, without the coefficients, would succeed each other in the following order; a^{10}, $a^9 b$, $a^8 b^2$, $a^7 b^3$, $a^6 b^4$, $a^5 b^5$, $a^4 b^6$, $a^3 b^7$, $a^2 b^8$, $a b^9$, b^{10}.

304. It remains, therefore, to show how we are to determine the coefficients which belong to those terms, or the numbers by which they are to be multiplied. Now, *with respect to the first term, its coefficient is always unity; and with regard to the second, its coefficient is constantly the exponent of the power;* but with regard to the other terms, it is not so easy to observe any order in their coefficients. However, if we continue those coefficients, we shall not fail to discover a law, by which we may advance as far as we please. This the following table will show.

Powers.	Coefficients.
I.	1, 1
II.	1, 2, 1
III.	1, 3, 3, 1
IV.	1, 4, 6, 4, 1
V.	1, 5, 10, 10, 5, 1
VI.	1, 6, 15, 20, 15, 6, 1
VII.	1, 7, 21, 35, 35, 21, 7, 1
VIII.	1, 8, 28, 56, 70, 56, 28, 8, 1
IX.	1, 9, 36, 84, 126, 126, 84, 36, 9, 1
X.	1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1, &c.

We see then, that the tenth power of $a + b$ will be $a^{10} + 10 a^9 b + 45 a^8 bb + 120 a^7 b^3 + 210 a^6 b^4 + 252 a^5 b^5 + 210 a^4 b^6 + 120 a^3 b^7 + 45 aa b^8 + 10 a b^9 + b^{10}$.

305. *With regard to the coefficients, it must be observed, that for each power their sum must be equal to the number 2 raised to the same power.* Let $a = 1$ and $b = 1$, each term, without the coefficients, will be $= 1$; consequently, the value of the power will be simply the sum of the coefficients; this sum, in the preceding example, is 1024, and accordingly

$$(1 + 1)^{10} = 2^{10} = 1024.$$

It is the same with respect to other powers; we have for the

I. $1 + 1 = 2 = 2^1$,
II. $1 + 2 + 1 = 4 = 2^2$,
III. $1 + 3 + 3 + 1 = 8 = 2^3$,
IV. $1 + 4 + 6 + 4 + 1 = 16 = 2^4$,
V. $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$,
VI. $1 + 6 + 15 + 20 + 15 + 6 + 1 = 64 = 2^6$,
VII. $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128 = 2^7$,
&c.

306. Another necessary remark, with regard to the coefficients, is, that they increase from the beginning to the middle, and then decrease in the same order. In the even powers, the greatest coefficient is exactly in the middle; but in the odd powers, two coefficients, equal and greater than the others, are found in the middle, belonging to the mean terms.

The order of the coefficients deserves particular attention; for it is in this order that we discover the means of determining them for any power whatever, without calculating all the preceding powers. We shall explain this method, reserving the demonstration however for the next chapter.

307. *In order to find the coefficients of any power proposed, the seventh, for example, let us write the following fractions, one after the other;*

$$\tfrac{7}{1},\ \tfrac{6}{2},\ \tfrac{5}{3},\ \tfrac{4}{4},\ \tfrac{3}{5},\ \tfrac{2}{6},\ \tfrac{1}{7}.$$

In this arrangement we perceive that the numerators begin by the exponent of the power required, and that they diminish successively by unity; while the denominators follow in the natural order of the numbers, 1, 2, 3, 4, *&c. Now, the first coefficient being always* 1, *the first fraction gives the second coefficient. The product of the two first fractions, multiplied together, represents the third coefficient. The product of the three first fractions represents the fourth coefficient, and so on.*

So that the first coefficient $= 1$; the second $= \frac{7}{1} = 7$; the third $= \frac{7}{1} \times \frac{6}{2} = 21$; the fourth $= \frac{7}{1} \times \frac{6}{2} \times \frac{5}{3} = 35$; the fifth $= \frac{7}{1} \times \frac{6}{2} \times \frac{5}{3} \times \frac{4}{4} = 35$; the sixth $= \frac{7}{1} \times \frac{6}{2} \times \frac{5}{3} \times \frac{4}{4} \times \frac{3}{5} = 21$; the seventh $= 21 \times \frac{2}{6} = 7$; the eighth $= 7 \times \frac{1}{7} = 1$.

308. So that we have, for the second power, the two fractions $\frac{2}{1}, \frac{1}{2}$; whence it follows, that the first coefficient $= 1$; the second $= \frac{2}{1} = 2$; and the third $= 2 \times \frac{1}{2} = 1$.

The third power furnishes the fractions $\frac{3}{1}, \frac{2}{2}, \frac{1}{3}$; wherefore the first coefficient $= 1$; the second $= \frac{3}{1} = 3$; the third $= 3 \times \frac{2}{2} = 3$; the fourth $= \frac{3}{1} \times \frac{2}{2} \times \frac{1}{3} = 1$.

We have for the fourth power, the fractions $\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}$; consequently the first coefficient $= 1$; the second $\frac{4}{1} = 4$; the third $\frac{4}{1} \times \frac{3}{2} = 6$; the fourth $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} = 4$; and the fifth $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4} = 1$.

309. This rule evidently renders it unnecessary for us to find the preceding coefficients, and enables us to discover immediately the coefficients which belong to any power. Thus, for the tenth power, we write the fractions $\frac{10}{1}, \frac{9}{2}, \frac{8}{3}, \frac{7}{4}, \frac{6}{5}, \frac{5}{6}, \frac{4}{7}, \frac{3}{8}, \frac{2}{9}, \frac{1}{10}$, by means of which we find

the first coefficient	$= 1$,
the second	$= \frac{10}{1} = 10$,
the third	$= 10 \times \frac{9}{2} = 45$,
the fourth	$= 45 \times \frac{8}{3} = 120$,
the fifth	$= 120 \times \frac{7}{4} = 210$,
the sixth	$= 210 \times \frac{6}{5} = 252$,
the seventh	$= 252 \times \frac{5}{6} = 210$,
the eighth	$= 210 \times \frac{4}{7} = 120$,
the ninth	$= 120 \times \frac{3}{8} = 45$,
the tenth	$= 45 \times \frac{2}{9} = 10$,
the eleventh	$= 10 \times \frac{1}{10} = 1$.

310. We may also write these fractions as they are, without computing their value; and in this way it is easy to express any power of $a + b$, however high. Thus, the hundredth power of $a + b$, will be

$$(a+b)^{100} = a^{100} + \frac{100}{1} \times a^{99}b + \frac{100 \times 99}{1 \times 2} + a^{98}b^2 + \frac{100 \times 99 \times 98}{1 \times 2 \times 3} a^{97}b^3 + \frac{100 \times 99 \times 98 \times 97}{1 \times 2 \times 3 \times 4} a^{96}b^4 +, \&c.,$$

whence the law of the succeeding terms may be easily deduced.

CHAPTER XI.

Of the Transposition of the Letters, on which the Demonstration of the preceding Rule is founded.

311. If we trace back the origin of the coefficients which we have been considering, we shall find, that each term is presented, as many times as it is possible to transpose the letters, of which that term consists; or, to express the same thing differently, the coefficient of each term is equal to the number of transpositions that the letters admit, of which that term is composed. In the second power, for example, the term ab is taken twice, that is to say, its coefficient is 2; and in fact we may change the order of the letters which compose that term twice, since we may write ab and ba; the term aa, on the contrary, is found only once, because the order of the letters can undergo no change or transposition. In the third power of $a+b$, the term aab may be written in three different ways, aab, aba, baa; thus the coefficient is 3. Likewise, in the fourth power, the term a^3b or $aaab$, admits of four different arrangements, $aaab$, $aaba$, $abaa$, $baaa$; therefore its coefficient is 4. The term $aabb$, admits of six transpositions, $aabb$, $abba$, $baba$, $abab$, $bbaa$, $baab$, and its coefficient is 6. It is the same in all cases.

312. In fact, if we consider that the fourth power, for example, of any root consisting of more than two terms, as $(a+b+c+d)^4$, is found by multiplying the four factors, I. $a+b+c+d$; II. $a+b+c+d$; III. $a+b+c+d$; IV. $a+b+c+d$; we may easily see, that each letter of the first factor must be multiplied by each letter of the second, then by each letter of the third, and lastly, by each letter of the fourth.

Each term must therefore not only be composed of four letters, but also present itself, or enter into the sum, as many times as those letters can be differently arranged with respect to each other, whence arises its coefficient.

313. It is therefore of great importance to know, in how many different ways a given number of letters may be arranged. And, in this inquiry, we must particularly consider, whether the letters in question are the same, or different. When they are the same, there can be no transposition of them, and for this reason the simple powers, as a^2, a^3, a^4, &c., all have unity for the coefficient.

314. Let us first suppose all the letters different; and beginning with the simplest case of two letters, or $a\,b$, we immediately discover that two transpositions may take place, namely, $a\,b$, and $b\,a$.

If we have three letters, $a\,b\,c$, to consider, we observe that each of the three may take the first place, while the two others will admit of two transpositions. For if a is the first letter, we have two arrangements, $a\,b\,c$, $a\,c\,b$; if b is in the first place, we have the arrangements, $b\,a\,c$, $b\,c\,a$; lastly, if c occupies the first place, we have also two arrangements, namely, $c\,a\,b$, $c\,b\,a$. And consequently the whole number of arrangements is $3 \times 2 = 6$.

If there are four letters, $a\,b\,c\,d$, each may occupy the first place; and in each case the three others may form six different arrangements, as we have just seen. The whole number of transpositions is therefore $4 \times 6 = 24 = 4 \times 3 \times 2 \times 1$.

If there are five letters, $a\,b\,c\,d\,e$, each of the five must be the first, and the four others will admit of twenty-four transpositions; so that the whole number of transpositions will be

$$5 \times 24 = 120 = 5 \times 4 \times 3 \times 2 \times 1.$$

315. Consequently, however great the number of letters may be, it is evident, provided they are all different, that we may easily determine the number of transpositions, and that we may make use of the following table:

Number of Letters.	Number of Transpositions.
I.	$1 = 1.$
II.	$2 \times 2 = 2.$
III.	$3 \times 2 \times 1 = 6.$
IV.	$4 \times 3 \times 2 \times 1 = 24.$
V.	$5 \times 4 \times 3 \times 2 \times 1 = 120.$
VI.	$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$
VII.	$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040.$
VIII.	$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320.$
IX.	$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880.$
X.	$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3628800.$

316. But, as we have intimated, the numbers in this table can be made use of only when the letters are different; for if two or more of them are alike, the number of transpositions becomes much less; and if all the letters are the same, we have only one arrangement. We shall now see how the numbers in the table are to be diminished, according to the number of letters that are alike.

317. When two letters are given, and those letters are the same, the two arrangements are reduced to one, and consequently the number, which we have found above, is reduced to the half; that is to say, it must be divided by 2. If we have three letters alike, the six transpositions are reduced to one; whence it follows that the numbers in the table must be divided by $6 = 3 \times 2 \times 1$. And for the same reason, if four letters are alike, we must divide the numbers found by 24 or $4 \times 3 \times 2 + 1$, &c.

It is easy therefore to determine how many transpositions the letters $a\,a\,a\,b\,b\,c$, for example, may undergo. They are in number 6, and consequently, if they were all different, they would admit of $6 \times 5 \times 4 \times 3 \times 2 \times 1$ transpositions. But since a is found thrice in those letters, we must divide that number of transpositions, by $3 \times 2 \times 1$; and since b occurs twice, we must again divide it by 2×1; the number of transpositions required will therefore be $= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 2 \times 1 \times 2 \times 1} = 5 \times 4 \times 3 = 60$.

318. It will now be easy for us to determine the coefficients of all the terms of any power. We shall give an example of the seventh power $(a + b)^7$.

The first term is a^7, which occurs only once; and as all the other terms have each seven letters, it follows that the number of transpositions for each term would be $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, if all the letters were different. But since in the second term, $a^6 b$, we find six letters alike, we must divide the above product by $6 \times 5 \times 4 \times 3 \times 2 \times 1$, whence it follows that the coefficient is $= \frac{7 + 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{7}{1}$.

In the third term $a^5\,b\,b$, we find the same letter a five times, and the same letter b twice; we must therefore divide that number first by $5 \times 4 \times 3 \times 2 \times 1$, and then also by 2×1; whence results the coefficient $\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1} = \frac{7 \times 6}{2 \times 1}$.

The fourth term $a^4\,b^3$ contains the letter a four times, and the letter b thrice; consequently, the whole number of the transpositions of the seven letters must be divided, in the first place, by $4 \times 3 \times 2 \times 1$, and secondly, by $3 \times 2 \times 1$, and the coefficient becomes $= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} = \frac{7 \times 6 \times 5}{1 \times 2 \times 3}$.

In the same manner we find $\frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4}$ for the coefficient of the fifth term, and so of the rest; by which the rule before given is demonstrated.

319. These considerations carry us further, and show us also how to find all the powers of roots composed of more than two terms. We shall apply them to the third power of $a + b + c$; the terms of which must be formed by all the possible combinations of three letters, each term having for its coefficient the number of its transpositions, as above.

Without performing the multiplication, the third power of $(a + b + c)$ will be $a^3 + 3aab + 3aac + 3abb + 6abc + 3acc + b^3 + 3bbc + 3bcc + c^3$.

Suppose $a = 1$, $b = 1$, $c = 1$, the cube of $1 + 1 + 1$, or of 3, will be $1 + 3 + 3 + 3 + 6 + 3 + 1 + 3 + 3 + 1 = 27$.

This result is accurate, and confirms the rule.

If we had supposed $a = 1$, $b = 1$, and $c = -1$, we should have found for the cube of $1 + 1 - 1$, that is, of 1,

$$1 + 3 - 3 + 3 - 6 + 3 + 1 - 3 + 3 - 1 = 1.$$

CHAPTER XII.

Of the Expression of Irrational powers by Infinite Series.

320. As we have shown the method of finding any power of the root $a + b$, however great the exponent, we are able to express generally, the power of $a + b$, whose exponent is undetermined. It is evident that if we represent that exponent by n, we shall have by the rule already given (art. 307 and the following):

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1} \times \frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n-3}{4}a^{n-4}b^4, \text{ \&c.}$$

321. If the same power of the root $a - b$ were required, we should only change the signs of the second, fourth, sixth, &c. terms, and should have

$$(a-b)^n = a^n - \frac{n}{1}a^{n-1}b + \frac{n}{1}\times\frac{n-1}{2}a^{n-2}b^2 - \frac{n}{1}\times\frac{n-1}{2}\times$$
$$\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}a^{n-4}b^4, \text{ \&c.}$$

322. These formulas are remarkably useful; for they serve also to express all kinds of radicals. We have shown that all irrational quantities may assume the form of powers, whose exponents are fractional, and that $\sqrt[2]{a} = a^{\frac{1}{2}}$; $\sqrt[3]{a} = a^{\frac{1}{3}}$, and $\sqrt[4]{a} = a^{\frac{1}{4}}$, &c. We have therefore, also,

$$\sqrt[2]{(a+b)} = (a+b)^{\frac{1}{2}};\ \sqrt[3]{(a+b)} = (a+b)^{\frac{1}{3}}$$

and $$\sqrt[4]{(a+b)} = (a+b)^{\frac{1}{4}}, \text{ \&c.}$$

Wherefore, if we wish to find the square root of $a+b$, we have only to substitute for the exponent n the fraction $\frac{1}{2}$, in the general formula [art. 320], and we shall have for the first, for the coefficients,

$$\frac{n}{1}=\frac{1}{2};\ \frac{n-1}{2}=-\frac{1}{4};\ \frac{n-2}{3}=-\frac{3}{6};\ \frac{n-3}{4}=-\frac{5}{8};$$
$$\frac{n-4}{5}=-\frac{7}{10};\ \frac{n-5}{6}=-\frac{9}{12}.$$

Then

$$a^n = a^{\frac{1}{2}} = \sqrt{a} \text{ and } a^{n-1} = \frac{1}{\sqrt{a}};\ a^{n-2} = \frac{1}{a\sqrt{a}};\ a^{n-3} = \frac{1}{aa\sqrt{a}},$$

&c., or we might express those powers of a in the following manner;

$$a^n = \sqrt{a};\ a^{n-1} = \frac{a^n}{a} = \frac{\sqrt{a}}{a};\ a^{n-2} = \frac{a^n}{a^2} = \frac{\sqrt{a}}{a^2};$$
$$a^{n-3} = \frac{a^n}{a^3} = \frac{\sqrt{a}}{a^3};\ a^{n-4} = \frac{a^n}{a^4} = \frac{\sqrt{a}}{a^4}, \text{ \&c.}$$

323. This being laid down, the square root of $a+b$ may be expressed in the following manner:

$$\sqrt{(a+b)} =$$
$$\sqrt{a} + \frac{1}{2}b\frac{\sqrt{a}}{a} - \frac{1}{2}\times\frac{1}{4}bb\frac{\sqrt{a}}{aa} + \frac{1}{2}\times\frac{1}{4}\times\frac{3}{6}b^3\frac{\sqrt{a}}{a^3} - \frac{1}{2}\times\frac{1}{4}$$
$$\times\frac{3}{6}\times\frac{5}{8}b^4\frac{\sqrt{a}}{a^4}, \text{ \&c.}$$

324. If a, therefore, be a square number, we may assign the value of $\sqrt{a}$, and consequently, the square root of $a+b$ may be expressed by an infinite series, without any radical sign.

Let, for example, $a = cc$, we shall have $\sqrt{a} = c$; then

$$\sqrt{(cc+b)} = c + \frac{1}{2}\times\frac{b}{c} - \frac{1}{8}\frac{bb}{c^3} + \frac{1}{16}\times\frac{b^3}{c^5} - \frac{5}{128}\times\frac{b^4}{c^7}, \text{ \&c.}$$

We see, therefore, that there is no number, whose square root we may not extract in the same way; since every number may be resolved into two parts, one of which is a square represented by cc. If we require, for example, the square root of 6, we make $6 = 4 + 2$, consequently $cc = 4$, $c = 2$, $b = 2$, whence results

$$\sqrt{6} = 2 + \tfrac{1}{2} - \tfrac{1}{16} + \tfrac{1}{64} - \tfrac{5}{1024}, \text{ \&c.}$$

If we take only the leading terms of this series, we shall have $2\tfrac{1}{2} = \tfrac{5}{2}$, the square of which, $\tfrac{25}{4}$, is $\tfrac{1}{4}$ greater than 6; but if we consider three terms, we have $2\tfrac{7}{16} = \tfrac{39}{16}$, the square of which, $\tfrac{1521}{256}$, is still $\tfrac{15}{256}$ too small.

325. Since, in this example, $\tfrac{5}{2}$ approaches very nearly to the true value of $\sqrt{6}$, we shall take for 6 the equivalent quantity $\tfrac{25}{4} - \tfrac{1}{4}$. Thus $cc = \tfrac{25}{4}$; $c = \tfrac{5}{2}$; $b = -\tfrac{1}{4}$; and calculating only the two leading terms, we find

$$\sqrt{6} = \tfrac{5}{2} + \tfrac{1}{2}\times\frac{-\tfrac{1}{4}}{\tfrac{5}{2}} = \tfrac{5}{2} - \tfrac{1}{2}\times\frac{\tfrac{1}{4}}{\tfrac{5}{2}} = \tfrac{5}{2} - \tfrac{1}{20} = \tfrac{49}{20}:$$

the square of this fraction being $\tfrac{2401}{400}$, exceeds the square of $\sqrt{6}$ only by $\tfrac{1}{400}$.

Now, making $6 = \tfrac{2401}{400} - \tfrac{1}{400}$, so that $c = \tfrac{49}{20}$ and $b = -\tfrac{1}{400}$; and still taking only the two leading terms, we have

$$\sqrt{6} = \tfrac{49}{20} + \tfrac{1}{2}\times\frac{-\tfrac{1}{400}}{\tfrac{49}{20}} = \tfrac{49}{20} - \tfrac{1}{2}\times\frac{\tfrac{1}{400}}{\tfrac{49}{20}} = \tfrac{49}{20} - \tfrac{1}{1960} = \tfrac{4801}{1960},$$

the square of which is $\tfrac{23049601}{3841600}$. Now 6, when reduced to the same denominator, is $= \tfrac{23049600}{3841600}$; the error therefore is only $\tfrac{1}{3841600}$.

326. In the same manner, we may express the cube root of $a + b$ by an infinite series. For since $\sqrt[3]{(a+b)} = (a+b)^{\frac{1}{3}}$ we shall have in the general formula $n = \tfrac{1}{3}$, and for the coefficients,

$$\frac{n}{1} = \frac{1}{3};\ \frac{n-1}{2} = -\frac{1}{3};\ \frac{n-2}{3} = -\frac{5}{9};\ \frac{n-3}{4} = -\frac{2}{3};$$

$$\frac{n-4}{5} = -\frac{11}{15}, \text{ \&c.}$$

and with regard to the powers of a, we shall have

$$a^n = \sqrt[3]{a};\ a^{n-1} = \frac{\sqrt[3]{a}}{a};\ a^{n-2} = \frac{\sqrt[3]{a}}{a\,a};\ a^{n-3} = \frac{\sqrt[3]{a}}{a^3},\ \&c.;$$

then

$$\sqrt[3]{(a+b)} = \sqrt[3]{a} + \frac{1}{3} \times b\frac{\sqrt[3]{a}}{a} - \frac{1}{9} \times bb\frac{\sqrt[3]{a}}{a\,a} + \frac{5}{81} \times b^3\frac{\sqrt[3]{a}}{a^3} - \frac{10}{243} \times b^4\frac{\sqrt[3]{a}}{a^4},\ \&c.$$

327. If a therefore be a cube, or $a = c^3$, we have $\sqrt[3]{a} = c$, and the radical signs will vanish; for we shall have

$$\sqrt[3]{(c^3+b)} = c + \frac{1}{3} \times \frac{b}{c\,c} - \frac{1}{9} \times \frac{bb}{c^5} + \frac{5}{81} \times \frac{b^3}{c^8} - \frac{10}{243} \times \frac{b^4}{c^{11}},\ \&c.$$

328. We have, therefore, arrived at a formula, which will enable us to find *by approximation*, as it is called, the cube root of any number; since every number may be resolved into two parts, as $c^3 + b$, the first of which is a cube.

If we wish, for example, to determine the cube root of 2, we represent 2 by $1 + 1$, so that $c = 1$, and $b = 1$, consequently $\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{1}{9} + \frac{5}{81}$, &c., the two leading terms of this series make $1\frac{1}{3} = \frac{4}{3}$ the cube of which, $\frac{64}{27}$, is too great by $\frac{10}{27}$. Let us then make $2 = \frac{64}{27} - \frac{10}{27}$, we have $c = \frac{4}{3}$ and $b = -\frac{10}{27}$, and consequently $\sqrt[3]{2} = \frac{4}{3} + \frac{1}{3} \times \frac{-\frac{10}{27}}{\frac{16}{9}}$. These two terms give

$$\frac{4}{3} - \frac{5}{72} = \frac{91}{72},\ \text{the cube of which is}\ \frac{753571}{373248}.$$

Now, $2 = \frac{746496}{373248}$, so that the error is $\frac{7075}{373248}$. In this way we might still approximate, and the faster in proportion as we take a greater number of terms.

CHAPTER XIII.

Of the Resolution of Negative Powers.

329. We have already shown, that we may express $\frac{1}{a}$ by a^{-1}; we may therefore also express $\frac{1}{a+b}$ by $(a+b)^{-1}$; so that the

fraction $\frac{1}{a+b}$ may be considered as a power of $a+b$, namely, that power whose exponent is -1; and from this it follows, that the series already found as the value of $(a+b)^n$ extends also to this case.

330. Since, therefore, $\frac{1}{a+b}$ is the same as $(a+b)^{-1}$, let us suppose, in the general formula, $n=-1$; and we shall first have for the coefficients

$$\frac{n}{1}=-1;\ \frac{n-1}{2}=-1;\ \frac{n-2}{3}=-1;\ \frac{n-3}{4}=-1,\ \&c.$$

Then, for the powers of a;

$$a^n=a^{-1}=\frac{1}{a};\ a^{n-1}=a^{-2}=\frac{1}{a^2};\ a^{n-2}=\frac{1}{a^3};\ a^{n-3}=\frac{1}{a^4},\ \&c.$$

So that

$$(a+b)^{-1}=\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{bb}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}-\frac{b^5}{a^6},\ \&c.$$

and this is the same series that we found before by division.

331. Further, $\frac{1}{(a+b)^2}$ being the same with $(a+b)^{-2}$, let us reduce this quantity also to an infinite series. For this purpose, we must suppose $n=-2$, and we shall first have for the coefficients

$$\frac{n}{1}=-\frac{2}{1};\ \frac{n-1}{2}=-\frac{3}{2};\ \frac{n-2}{3}=-\frac{4}{3};\ \frac{n-3}{4}=-\frac{5}{4},\ \&c.$$

Then, for the powers of a;

$$a^n=\frac{1}{a^2};\ a^{n-1}=\frac{1}{a^3};\ a^{n-2}=\frac{1}{a^4};\ a^{n-3}=\frac{1}{a^5},\ \&c.$$

We therefore obtain

$$(a+b)^{-2}=\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2}{1}\times\frac{b}{a^3}+\frac{2}{1}\times\frac{3}{2}\times\frac{bb}{a^4}-\frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times\frac{b^3}{a^5}+\frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\frac{b^4}{a^6},\ \&c.$$

Now,

$$\tfrac{2}{1}=2;\ \tfrac{2}{1}\times\tfrac{3}{2}=3;\ \tfrac{2}{1}\times\tfrac{3}{2}\times\tfrac{4}{3}=4;\ \tfrac{2}{1}\times\tfrac{3}{2}\times\tfrac{4}{3}\times\tfrac{5}{4}=5,\ \&c.$$

Consequently we have

$$\frac{1}{(a+b)^2}=\frac{1}{a^2}-2\frac{b}{a^3}+3\frac{b^2}{a^4}-4\frac{b^3}{a^5}+5\frac{b^4}{a^6}-6\frac{b^5}{a^7}+7\frac{b^6}{a^8},$$

&c.

332. Let us proceed and suppose $n = -3$, and we shall have a series expressing the value of $\frac{1}{(a+b)^3}$, or of $(a+b)^{-3}$. The coefficients will be

$$\frac{n}{1} = -\frac{3}{1};\ \frac{n-1}{2} = -\frac{4}{2};\ \frac{n-2}{3} = -\frac{5}{3};\ \frac{n-3}{4} = -\frac{6}{4},\ \&c.$$

and the powers of a become,

$$a^n = \frac{1}{a^3};\ a^{n-1} = \frac{1}{a^4};\ a^{n-2} = \frac{1}{a^5},\ \&c.$$

which gives

$$\frac{1}{(a+b)^3} = \frac{1}{a^3} - \frac{3b}{1a^4} + \frac{3}{1} \times \frac{4b^2}{2a^5} - \frac{3}{1} \times \frac{4}{2} \times \frac{5b^3}{3a^6} + \frac{3}{1} \times \frac{4}{2} \times \frac{5}{3} \times \frac{6b^4}{4a^7},\ \&c.$$

$$= \frac{1}{a^3} - 3\frac{b}{a^4} + 6\frac{b^2}{a^5} - 10\frac{b^3}{a^6} + 15\frac{b^4}{a^7} - 21\frac{b^5}{a^8} + 28\frac{b^6}{a^9} - 36\frac{b^7}{a^{10}} + 45\frac{b^8}{a^{11}},\ \&c.$$

Let us now make $n = -4$; we shall have for the coefficients

$$\frac{n}{1} = -\frac{4}{1};\ \frac{n-1}{2} = -\frac{5}{2};\ \frac{n-2}{3} = -\frac{6}{3};\ \frac{n-3}{4} = -\frac{7}{4},\ \&c.,$$

and for the powers,

$$a^n = \frac{1}{a^4};\ a^{n-1} = \frac{1}{a^5};\ a^{n-2} = \frac{1}{a^6};\ a^{n-3} = \frac{1}{a^7};\ a^{n-4} = \frac{1}{a^8},\ \&c.,$$

whence we obtain,

$$\frac{1}{(a+b)^4} = \frac{1}{a^4} - \frac{4}{1} \times \frac{b}{a^5} + \frac{4}{1} \times \frac{5}{2} \times \frac{b^2}{a^6} - \frac{4}{1} \times \frac{5}{2} \times \frac{6}{3} \times \frac{b^3}{a^7} + \frac{4}{1} \times \frac{5}{2} \times \frac{6}{3} \times \frac{7}{4} \times \frac{b^4}{a^8},\ \&c.$$

$$= \frac{1}{a^4} - 4\frac{b}{a^5} - 10\frac{b^2}{a^6} - 20\frac{b^3}{a^7} + 35\frac{b^4}{a^8} - 56\frac{b^5}{a^9},\ \&c.$$

333. The different cases that have been considered enable us to conclude with certainty, that we shall have, generally, for any negative power of $a + b$;

$$\frac{1}{(a+b)^m} = \frac{1}{a^m} - \frac{m}{1} \times \frac{b}{a^{m+1}} + \frac{m}{1} \times \frac{m+1}{2} \times \frac{b^2}{a^{m+2}} - \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{b^3}{a^{m+3}}, \text{ \&c.}$$

And by means of this formula we may transform all such fractions into infinite series, substituting fractions also, or fractional exponents, for m, in order to express irrational quantities.

334. The following considerations will illustrate this subject further.

We have seen that,

$$\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} - \frac{b^5}{a^6}, \text{ \&c.}$$

If, therefore, we multiply this series by $a + b$, the product ought to be $= 1$; and this is found to be true, as we shall see by performing the multiplication:

$$\begin{array}{l} \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} - \frac{b^5}{a^6} +, \text{ \&c.} \\ a + b \\ \hline 1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{b^3}{a^3} + \frac{b^4}{a^4} - \frac{b^5}{a^5}, +, \text{ \&c.} \\ \quad + \frac{b}{a} - \frac{b^2}{a^2} + \frac{b^3}{a^3} - \frac{b^4}{a^4} + \frac{b^5}{a^5}, -, \text{ \&c.} \\ \hline 1. \end{array}$$

335. We have also found, that

$$\frac{1}{(a+b)^2} = \frac{1}{aa} - \frac{2b}{a^3} + \frac{3bb}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6} - \frac{6b^5}{a^7}, \text{ \&c.}$$

If, therefore, we multiply this series by $(a + b)^2$, the product ought also to be $= 1$. Now $(a + b)^2 = aa + 2ab + bb$. See the operation:

$$\frac{1}{aa} - \frac{2b}{a^3} + \frac{3bb}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6} - \frac{6b^5}{a^7} +, \&c.$$
$$aa + 2ab + bb$$

$$1 - \frac{2b}{a} + \frac{3bb}{aa} - \frac{4b^3}{a^3} + \frac{5b^4}{a^4} - \frac{6b^5}{a^5} +, \&c.$$
$$+ \frac{2b}{a} - \frac{4bb}{aa} + \frac{6b^3}{a^3} - \frac{8b^4}{a^4} + \frac{10b^5}{a^5} -, \&c.$$
$$+ \frac{bb}{aa} - \frac{2b^3}{a^3} + \frac{3b^4}{a^4} - \frac{4b^5}{a^5} +, \&c.$$

1 = the product, which the nature of the thing required.

336. If we multiply the series which we found for the value of $\frac{1}{(a+b)^2}$, by $a + b$ only, the product ought to answer to the fraction $\frac{1}{a+b}$, or be equal to the series already found, namely,

$$\frac{1}{a} - \frac{b}{a^2} + \frac{bb}{a^3} - \frac{b^3}{a^4} + \frac{b^5}{a^5}, \&c.$$

and this the actual multiplication will confirm.

$$\frac{1}{aa} - \frac{2b}{a^3} + \frac{3bb}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6}, \&c.$$
$$a + b$$

$$\frac{1}{a} - \frac{2b}{aa} + \frac{3bb}{a^3} - \frac{4b^3}{a^4} + \frac{5b^4}{a^5}, \&c.$$
$$+ \frac{b}{aa} - \frac{2bb}{a^3} + \frac{3b^3}{a^4} - \frac{4b^4}{a^5}, \&c.$$

$$\frac{1}{a} - \frac{b}{aa} + \frac{bb}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} -, \&c.$$

SECTION III.

OF RATIOS AND PROPORTIONS.

CHAPTER I.

Of Arithmetical Ratio, or of the Difference between Two Numbers.

ARTICLE 337. Two quantities are either equal to one another, or they are not. In the latter case, where one is greater than the other, we may consider their inequality in two different points of view: we may ask, *how much* one of the quantities is greater than the other? Or we may ask, *how many times* the one is greater than the other? The results, which constitute the answers to these two questions, are both called *relations* or *ratios*. We usually call the former *arithmetical ratio*, and the latter *geometrical ratio*, without however these denominations having any connexion with the thing itself: they have been adopted arbitrarily.

338. It is evident that the quantities of which we speak must be of one and the same kind; otherwise we could not determine any thing with regard to their equality or inequality. It would be absurd, for example, to ask if two pounds and three ells are equal quantities. So that, in what follows, quantities of the same kind only are to be considered; and as they may always be expressed by numbers, it is of numbers only, as was mentioned at the beginning, that we shall treat.

339. When of two given numbers, therefore, it is required to find how much one is greater than the other, the answer to this question determines the arithmetical ratio of the two numbers. Now, since this answer consists in giving the difference of the two numbers, it follows that an arithmetical ratio is nothing but the *difference* be-

tween two numbers: and as this appears to be a better expression, we shall reserve the words *ratio* and *relation*, to express geometrical ratios.

340. The difference between two numbers is found, we know, by subtracting the less from the greater; nothing therefore can be easier than resolving the question, how much one is greater than the other. So that when the numbers are equal, the difference being nothing, if it be inquired how much one of the numbers is greater than the other, we answer, By nothing. For example, 6 being $= 2 \times 3$, the difference between 6 and 2×3 is 0.

341. But when the two numbers are not equal, as 5 and 3, and it is inquired how much 5 is greater than 3, the answer is 2; and it is obtained by subtracting 3 from 5. Likewise 15 is greater than 5 by 10; and 20 exceeds 8 by 12.

342. We have three things, therefore, to consider on this subject; 1st, the greater of the two numbers; 2d, the less; and 3d, the difference. And these three quantities are connected together in such a manner, that two of the three being given, we may always determine the third.

Let the greater number $= a$, the less $= b$, and the difference $= d$; the difference d will be found by subtracting b from a, so that $d = a - b$; whence we see how to find d, when a and b are given.

343. But if the difference and the less of the two numbers or b, are given, we can determine the greater number by adding together the difference and the less number, which gives $a = b + d$. For, if we take from $b + d$ the less number b, there remains d, which is the known difference. Let the less number = 12, and the difference = 8; then the greater number will be = 20.

344. Lastly, if beside the difference d, the greater number a is given, the other number b is found by subtracting the difference from the greater number, which gives $b = a - d$. For if I take the number $a - d$ from the greater number a, there remains d, which is the given difference.

345. The connexion, therefore, among the numbers a, b, d, is of such a nature as to give the three following results: 1st. $d = a - b$; 2d. $a = b + d$; 3d. $b = a - d$; and if one of these three comparisons be just, the others must necessarily be so also. Wherefore, generally, if $z = x + y$, it necessarily follows, that $y = z - x$, and $x = z - y$.

346. With regard to these arithmetical ratios we must remark, that *if we add to the two numbers* a *and* b, *a number* c *assumed at pleasure, or subtracted from them, the difference remains the same.* That is to say, if d is the difference between a and b, that number d will also be the difference between $a + c$ and $b + c$, and between $a - c$ and $b - c$. For example, the difference between the numbers 20 and 12 being 8, that difference will remain the same, whatever number we add to the numbers 20 and 12, and whatever numbers we subtract from them.

347. The proof is evident; for if $a - b = d$ we have also $(a + c) - (b + c) = d$; and also $(a - c) - (b - c) = d$.

348. *If we double the two numbers* a *and* b, *the difference will also become double.* Thus, when $a - b = d$, we shall have, $2a - 2b = 2d$; and, *generally*, $na - nb = nd$, *whatever value we give to* n.

CHAPTER II.

Of Arithmetical Proportion.

349. When two arithmetical ratios, or relations, are equal, this equality is called an *arithmetical proportion.*

Thus, when $a - b = d$ and $p - q = d$, so that the difference is the same between the numbers p and q, as between the numbers a and b, we say that these four numbers form an arithmetical proportion; which we write thus, $a - b = p - q$, expressing clearly by this, that the difference between a and b is equal to the difference between p and q.

350. An arithmetical proportion consists therefore of four terms, which must be such, that if we subtract the second from the first, the remainder is the same as when we subtract the fourth from the third. Thus, the four numbers 12, 7, 9, 4, form an arithmetical proportion, because $12 - 7 = 9 - 4$.*

* To show that these terms make such a proportion, some write them thus; 12..7::9..4.

351. ***When we have an arithmetical proportion, as*** $a - b = p - q$, ***we may make the second and third change places, writing***

$$a - p = b - q;$$

and this equality will be no less true; for, since $a - b = p - q$, add b to both sides, and we have $a = b + p - q$; then subtract p from both sides, and we have $a - p = b - q$.

In the same manner, as $12 - 7 = 9 - 4$, so also

$$12 - 9 = 7 - 4.$$

352. *We may, in every arithmetical proportion, put the second term also in the place of the first, if we make the same transposition of the third and fourth.* That is to say, if $a - b = p - q$, we have also $b - a = q - p$. For $b - a$ is the negative of $a - b$, and $q - p$ is also the negative of $p - q$. Thus, since $12 - 7 = 9 - 4$, we have also $7 - 12 = 4 - 9$.

353. *But the great property of every arithmetical proportion is this; that the sum of the second and third term is always equal to the sum of the first and fourth.* This property, which we must particularly consider, is expressed also by saying that the sum of the *means* is equal to the sum of the *extremes*. Thus, since

$$12 - 7 = 9 - 4,$$

we have $7 + 9 = 12 + 4$; and the sum we find is 16 in both.

354. In order to demonstrate this principal property, let

$$a - b = p - q;$$

if we add to both $b + q$, we have $a + q = b + p$; that is, the sum of the first and fourth terms is equal to the sum of the second and third. *And conversely, if four numbers,* a, b, p, q, *are such, that the sum of the second and third is equal to the sum of the first and fourth,* that is, if $b + p = a + q$, we conclude, without a possibility of mistake, that *these numbers are in arithmetical proportion,* and that $a - b = p - q$. For, since $a + q = b + p$, if we subtract from both sides $b + q$, we obtain $a - b = p - q$.

Thus, the numbers 18, 13, 15, 10, being such, that the sum of the means ($13 + 15 = 28$), is equal to sum of the extremes ($18 + 10 = 28$), it is certain, that they also form an arithmetical proportion; and, consequently, that $18 - 13 = 15 - 10$.

355. It is easy, by means of this property, to resolve the following question. The three first terms of an arithmetical proportion

being given, to find the fourth? Let a, b, p, be the three first terms, and let us express the fourth by q, which it is required to determine, then $a + q = b + p$; by subtracting a from both sides, we obtain $q = b + p - a$.

Thus, *the fourth term is found by adding together the second and third, and subtracting the first from that sum.* Suppose, for example, that 19, 28, 13, are the three first terms given, the sum of the second and third is $= 41$; take from it the first, which is 19, there remains 22 for the fourth term sought, and the arithmetical proportion will be represented by $19 - 28 = 13 - 22$, or by

$$28 - 19 = 22 - 13,$$

or lastly, by $28 - 22 = 19 - 13$.

356. *When in an arithmetical proportion, the second term is equal to the third, we have only three numbers*; the property of which is this, that the first, minus the second, is equal to the second, minus the third; or, that the difference between the first and the second number is equal to the difference between the second and the third. The three numbers, 19, 15, 11, are of this kind, since

$$19 - 15 = 15 - 11.$$

357. Three *such numbers are said to form a continued arithmetical proportion,* which is sometimes written thus, 19 : 15 : 11. *Such proportions are also called* arithmetical progressions, *particularly if a greater number of terms follow each other according to the same law.*

An arithmetical progression may be either *increasing*, or *decreasing*. The former distinction is applied when the terms go on increasing, that is to say, when the second exceeds the first, and the third exceeds the second by the same quantity; as in the numbers 4, 7, 10. The decreasing progression is that, in which the terms go on always diminishing by the same quantity, such as the numbers 9, 5, 1.

358. Let us suppose the numbers a, b, c, to be in arithmetical progression; then $a - b = b - c$, whence it follows, from the equality between the sum of the extremes and that of the means, that $2b = a + c$; and if we subtract a from both, we have

$$c = 2b - a.$$

359. So that *when the two first terms,* a, b, *of an arithmetical progression are given, the third is found by taking the first from*

twice the second. Let 1 and 3 be the two first terms of an arithmetical progression, the third will be $= 2 \times 3 - 1 = 5$. And these three numbers, 1, 3, 5, give the proportion $1 - 3 = 3 - 5$.

360. By following the same method, we may pursue the arithmetical progression as far as we please; *we have only to find the fourth by means of the second and third, in the same manner* as we determined the third by means of the first and second, and so on. Let a be the first term, and b the second, the third will be $= 2b - a$, the fourth $= 4b - 2a - b = 3b - 2a$, the fifth

$$= 6b - 4a - 2b + a = 4b - 3a,$$

the sixth $= 8b - 6a - 3b + 2a = 5b - 4a$, the seventh

$$= 10b - 8a - 4b + 3a = 6b - 5a, \text{ \&c.}$$

CHAPTER III.

Of Arithmetical Progressions.

361. We have remarked already, that a series of numbers composed of any number of terms, which always increase, or decrease by the same quantity, is called an *arithmetical progression.*

Thus, the natural numbers written in their order, (as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c.) form an arithmetical progression, because they constantly increase by unity; and the series 25, 22, 19, 16, 13, 10, 7, 4, 1, &c. is also such a progression, since the numbers constantly decrease by 3.

362. The number, or quantity, by which the terms of an arithmetical progression become greater or less, is called the *difference.* So that when the first term and the difference are given, we may continue the arithmetical progression to any length.

For example, let the first term $= 2$, and the difference $= 3$, and we shall have the following increasing progression; 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, &c. in which each term is found, by adding the difference to the preceding term.

363. It is usual to write the natural numbers, 1, 2, 3, 4, 5, &c. above the terms of such an arithmetical progression, in order that

we may immediately perceive the rank which any term holds in the progression. These numbers, written above the terms, may be called *indices*; and the above example is written as follows:

Indices,	1	2	3	4	5	6	7	8	9	10
Arith. Prog.	2,	5,	8,	11,	14,	17,	20,	23,	26,	29, &c.

where we see that 29 is the tenth term.

364. Let a be the first term, and d the difference, the arithmetical progression will go on in the following order:

1	2	3	4	5	6	7	
a,	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$,	$a+5d$,	$a+6d$,	&c.

whence it appears, that any term of the progression might be easily found, without the necessity of finding all the preceding ones, by means only of the first term a, and the difference d. For example, the tenth term will be $= a + 9d$, the hundredth term $= a + 99d$, and generally, the term n will be $= a + (n - 1)d$.

365. When we stop at any point of the progression, it is of importance to attend to the first and the last term, since the index of the last will represent the number of terms. *If, therefore, the first term* $= a$, *the difference* $= d$, *and the number of terms* $= n$, *we shall have the last term* $= a + (n - 1)d$, *which is consequently found by multiplying the difference by the number of terms, minus one, and adding the first term to that product.* Suppose, for example, in an arithmetical progression of a hundred terms, the first term is $= 4$, and the difference $= 3$; then the last term will be

$$= 99 \times 3 + 4 = 301.$$

366. When we know the first term a and the last z, with the number of terms n, we can find the difference d. For, since the last term $z = a + (n - 1)d$, if we subtract a from both sides, we obtain $z - a = (n - 1)d$. So that by subtracting the first term from the last, we have the product of the difference multiplied by the number of terms *minus* 1. We have, therefore, only to divide $z - a$ by $n - 1$ to obtain the required value of the difference d which will be $= \frac{z - a}{n - 1}$. This result furnishes the following rule: *Subtract the first term from the last, divide the remainder by the number of terms minus* 1, *and the quotient will be the difference:* by means of which we may write the whole progression.

367. Suppose, for example, that we have an arithmetical progression of nine terms, whose first is $= 2$, and last $= 26$, and that

it is required to find the difference. We must subtract the first term, 2, from the last, 26, and divide the remainder, which is 24, by 9 — 1, that is, by 8; the quotient 3 will be equal to the difference required, and the whole progression will be

1	2	3	4	5	6	7	8	9
2,	5,	8,	11,	14,	17,	20,	23,	26.

To give another example, let us suppose, that the first term = 1, the last = 2, the number of terms = 10, and that the arithmetical progression, answering to these suppositions, is required; we shall immediately have for the difference $\frac{2-1}{10-1} = \frac{1}{9}$, and thence conclude that the progression is

1	2	3	4	5	6	7	8	9	10
1,	$1\frac{1}{9}$,	$1\frac{2}{9}$,	$1\frac{3}{9}$,	$1\frac{4}{9}$,	$1\frac{5}{9}$,	$1\frac{6}{9}$,	$1\frac{7}{9}$,	$1\frac{8}{9}$	2.

Another Example. Let the first term = $2\frac{1}{3}$, the last = $12\frac{1}{2}$, and the number of terms = 7; the difference will be

$$\frac{12\frac{1}{2}-2\frac{1}{3}}{7-1} = \frac{10\frac{1}{6}}{6} = \frac{61}{36} = 1\,\frac{25}{36},$$

and consequently the progression

1	2	3	4	5	6	7
$2\frac{1}{3}$,	$4\frac{1}{36}$,	$5\frac{13}{18}$,	$7\frac{5}{12}$,	$9\frac{1}{9}$,	$10\frac{29}{36}$,	$12\frac{1}{2}$

368. If now the first term a, the last term z, and the difference d, are given, we may from them find the number of terms n. For since $z - a = (n-1)\,d$, by dividing the two sides by d, we have $\frac{z-a}{d} = n - 1$. Now, n being greater by 1 than $n - 1$, we have $n = \frac{z-a}{d} + 1$; consequently, *the number of terms is found by dividing the difference between the first and the last term, or* $z - a$, *by the difference of the progression, and adding unity to the quotient,* $\frac{z-a}{d}$.

For example, let the first term = 4, the last = 100, and the difference = 12, the number of terms will be $\frac{100-4}{12} + 1 = 9$; and these nine terms will be

1	2	3	4	5	6	7	8	9
4	16,	28,	40,	52,	64,	76,	88	100,

If the first term $= 2$, the last $= 6$, and difference $= 1\frac{1}{3}$, the number of terms will be $\frac{4}{1\frac{1}{3}} + 1 = 4$; and these four terms will be,

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3\frac{1}{3}, & 4\frac{2}{3}, & 6. \end{array}$$

Again, let the first term $= 3\frac{1}{3}$, the last $= 7\frac{2}{3}$, and the difference $= 1\frac{4}{9}$, the number of terms will be $= \frac{7\frac{2}{3} - 3\frac{1}{3}}{1\frac{4}{9}} + 1 = 4$; which are,

$$3\frac{1}{3}, \quad 4\frac{7}{9}, \quad 6\frac{2}{9}, \quad 7\frac{2}{3}.$$

369. It must be observed, however, that as the number of terms is necessarily an integer, if we had not obtained such a number for n, in the examples of the preceding article, the questions would have been absurd.

Whenever we do not obtain an integral number for the value of $\frac{z-a}{d}$, it will be impossible to resolve the question; and consequently, in order that questions of this kind may be possible, $z - a$, must be divisible by d.

370. From what has been said, it may be concluded, that we have always four quantities, or things, to consider in arithmetical progression;

I. The first term a.
II. The last term z.
III. The difference d.
IV. The number of terms n.

And the relation of these quantities to each other are such, that if we know three of them, we are able to determine the fourth; for,

I. *If* a, d, *and* n *are known, we have* $z = a + (n-1)\,d$.
II. *If* z, d, *and* n *are known, we have* $a = z - (n-1)\,d$.
III. *If* a, z, *and* n *are known, we have* $d = \frac{z-a}{n-1}$.
IV. *If* a, z, *and* d *are known, we have* $n = \frac{z-a}{d} + 1$.

CHAPTER IV.

Of the Summation of Arithmetical Progressions.

371. It is often necessary also to find the sum of an arithmetical progression. This might be done by adding all the terms together; but as the addition would be very tedious, when the progression consisted of a great number of terms, a rule has been devised, by which the sum may be more readily obtained.

372. We shall first consider a particular given progression, such that the first term = 2, the difference = 3, the last term = 29, and the number of terms = 10;

1	2	3	4	5	6	7	8	9	10
2,	5,	8,	11,	14,	17,	20,	23,	26,	29.

We see, in this progression, that the sum of the first and the last term = 31; the sum of the second and the last but one = 31; the sum of the third and the last but two = 31, and so on; and thence we conclude that the sum of any two terms equally distant, the one from the first, and the other from the last term, is always equal to the sum of the first and the last term.

373. The reasons of this may be easily traced. For, if we suppose the first $= a$, the last $= z$, and the difference $= d$, the sum of the first and the last term is $= a + z$; and the second term being $= a + d$, and the last but one $= z - d$, the sum of these two terms is also $= a + z$. Further, the third term being $a + 2d$, and the last but two $= z - 2d$, it is evident that these two terms also, when added together make $a + z$. The demonstration may be easily extended to all the rest.

374. To determine, therefore, the sum of the progression proposed, let us write the same progression term by term, inverted, and add the corresponding terms together, as follows:

$$\begin{array}{l} 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 \\ 29 + 26 + 23 + 20 + 17 + 14 + 11 + 8 + 5 + 2 \\ \hline 31 + 31 + 31 + 31 + 31 + 31 + 31 + 31 + 31 + 31. \end{array}$$

This series of equal terms is evidently equal to twice the sum of the given progression; now the number of these equal terms is 10, as in the progression, and their sum, consequently,

$$= 10 \times 31 = 310.$$

So that, since this sum is twice the sum of the arithmetical progression, the sum required must be $= 155$.

375. If we proceed in the same manner, with respect to any arithmetical progression, the first term of which is $= a$, the last $= z$, and the number of terms $= n$; writing under the given progression the same progression inverted, and adding term to term, we shall have a series of n terms, each of which will be $= a + z$; the sum of this series will consequently be $= n\ (a + z)$, and it will be twice the sum of the proposed arithmetical progression; which therefore will be $= \frac{n\ (a + z)}{2}$.

376. This result furnishes an easy method of finding the sum of any arithmetical progression; and may be reduced to the following rule:

Multiply the sum of the first and the last term by the number of terms, and half the product will be the sum of the whole progression.

Or, which amounts to the same, multiply the sum of the first and the last term by half the number of terms.

Or, multiply half the sum of the first and the last term by the whole number of terms. Each of these enunciations of the rule will give the sum of the progression.

377. It may be proper to illustrate this rule by some examples.

First, let it be required to find the sum of the progression of the natural numbers, 1, 2, 3, &c, to 100. This will be, by the first rule,

$$= \frac{100 \times 101}{2} = 50 \times 101 = 5050.$$

If it were required to tell how many strokes a clock strikes in twelve hours; we must add together the numbers 1, 2, 3, &c. as far as 12; now this sum is found immediately

$$= \frac{12 \times 13}{2} = 6 \times 13 = 78.$$

If we wished to know the sum of the same progression continued to 1000, we should find it to be 500500; and the sum of this progression continued to 10000, would be 50005000.

378. *Another Question.* A person buys a horse, on condition that for the first nail he shall pay 5 halfpence, for the second 8, for

the third 11, and so on; always increasing 3 halfpence more for each following one; the horse having 32 nails, it is required to tell how much he will cost the purchaser.

In this question, it is required to find the sum of an arithmetical progression, the first term of which is 5, the difference = 3, and the number of terms = 32. We must therefore begin by determining the last term; we find it (by the rule in articles 365 and 370) $= 5 + 31 \times 3 = 98$. After which the sum required is easily found $= \frac{103 \times 32}{2} = 103 \times 16$; whence we conclude that the horse cost 1648 halfpence, or 3*l.* 8*s.* 8*d.*

379. Generally, let the first term be $= a$, the difference $= d$, and the number of terms $= n$; and let it be required to find, by means of these data, the sum of the whole progression. As the last term must be $= a + (n - 1)\,d$, the sum of the first and last will be $= 2\,a + (n - 1)\,d$. Multiplying this sum by the number of terms n, we have $2\,n\,a + n\,(n - 1)\,d$; the sum required therefore will be $= n\,a + \frac{n\,(n - 1)\,d}{2}$.

This formula, if applied to the preceding example, or to $a = 5$, $d = 3$, and $n = 32$, gives

$$5 \times 32 + \frac{32 \times 31 \times 3}{2} = 160 + 1488 = 1648;$$

the same sum that we obtained before.

380. If it be required to add together all the natural numbers from 1 to n, we have, for finding this sum, the first term $= 1$, the last term $= n$, and the number of terms $= n$; wherefore the sum required is $= \frac{n\,n + n}{2} = \frac{n\,(n + 1)}{2}$.

If we make $n = 1766$, the sum of all the numbers, from 1 to 1766, will be $= 883 \times 1767 = 1560261$.

381. *Let the progression of uneven numbers be proposed,* 1, 3, 5, 7, &c. continued to n terms, and let the sum of it be required:

Here the first term is $= 1$, the difference $= 2$, the number of terms $= n$; the last term will therefore be

$$= 1 + (n - 1)\,2 = 2\,n - 1,$$

and consequently the sum required $= n\,n$.

The whole therefore consists in multiplying the number of terms by itself. So that *whatever number of terms of this progression we* *add together, the sum will be always a square, namely, the square of the number of terms.* This we shall exemplify as follows:

Indices,	1	2	3	4	5	6	7	8	9	10, &c.
Progress.	1,	3,	5,	7,	9,	11,	13,	15,	17,	19, &c.
Sum,	1,	4,	9,	16,	25,	36,	49,	64,	81,	100, &c.

382. Let the first term be $= 1$, the difference $= 3$, and the number of terms $= n$; we shall have the progression 1, 4, 7, 10, &c. the last term of which will be $1 + (n - 1)\,3 = 3\,n - 2$; wherefore the sum of the first and the last term $= 3\,n - 1$, and consequently, the sum of this progression

$$= \frac{n\,(3\,n - 1)}{2} = \frac{3\,n\,n - n}{2}.$$

If we suppose $n = 20$, the sum will be $= 10 \times 59 = 590$.

383. Again, let the first term $= 1$, the difference $= d$, and the number of terms $= n$; then the last term will be $= 1 + (n - 1)\,d$. Adding the first, we have $2 + (n - 1)\,d$, and multiplying by the number of terms, we have $2\,n + n\,(n - 1)\,d$; whence we deduce the sum of the progression $= n + \frac{n\,(n - 1)\,d}{2}$.

We subjoin the following small table:

If $d = 1$, the sum is $= n + \frac{n\,(n - 1)}{2} = \frac{n\,n + n}{2}$

$d = 2$, $\quad = n + \frac{2\,n\,(n - 1)}{2} = n\,n$

$d = 3$, $\quad = n + \frac{3\,n\,(n - 1)}{2} = \frac{3\,n\,n - n}{2}$

$d = 4$, $\quad = n + \frac{4\,n\,(n - 1)}{2} = 2\,n\,n - n$

$d = 5$, $\quad = n + \frac{5\,n\,(n - 1)}{2} = \frac{5\,n\,n - 3\,n}{2}$

$d = 6$, $\quad = n + \frac{6\,n\,(n - 1)}{2} = 3\,n\,n - 2\,n$

$d = 7$, $\quad = n + \frac{7\,n\,(n - 1)}{2} = \frac{7\,n\,n - 5\,n}{2}$

$d = 8$, $\quad = n + \frac{8\,n\,(n - 1)}{2} = 4\,n\,n - 3\,n$

$d = 9$, $\quad = n + \frac{9\,n\,(n - 1)}{2} = \frac{9\,n\,n - 7\,n}{2}$

$d = 10$, $\quad = n + \frac{10\,n\,(n - 1)}{2} = 5\,n\,n - 4\,n.$

CHAPTER V.

Of Geometrical Ratio.

384. The *geometrical ratio* of two numbers is found by resolving the question, *how many times* is one of those numbers greater than the other? This is done by dividing one by the other; and the quotient, therefore, expresses the ratio required.

385. We have here three things to consider; 1st, the first of the two given numbers, which is called the *antecedent;* 2dly, the other number, which is called the *consequent*; 3dly, the ratio of the two numbers, or the quotient arising from the division of the antecedent by the consequent. For example, if the relation of the numbers 18 and 12 be required, 18 is the antecedent, 12 is the consequent, and the ratio will be $\frac{18}{12} = 1\frac{1}{2}$; whence we see, that the antecedent contains the consequent once and a half.

386. It is usual to represent geometrical relation by two points, placed one above the other, between the antecedent and the consequent. Thus $a:b$ means the geometrical relation of these two numbers, or the ratio of b to a.

We have already remarked that this sign is employed to represent division, and for this reason we make use of it here; because, in order to know the ratio, we must divide a by b. The relation expressed by this sign, is read simply, a is to b.

387. Relation therefore is expressed by a fraction, whose numerator is the antecedent, and whose denominator is the consequent. Perspicuity requires that this fraction should always be reduced to its lowest terms; which is done, as we have already shown, by dividing both the numerator and the denominator by their greatest common divisor. Thus, the fraction $\frac{18}{12}$ becomes $\frac{3}{2}$, by dividing both terms by 6.

388. So that relations only differ according as their ratios are different; and there are as many different kinds of geometrical relations as we can conceive different ratios.

The first kind is undoubtedly that in which the ratio becomes unity; this case happens when the two numbers are equal, as in $3:3$; $4:4$; $a:$; the ratio is here 1, and for this reason we call it the relation of equality.

Next follow those relations in which the ratio is another whole number; in $4:2$ the ratio is 2, and is called *double* ratio; in $12:4$ the ratio is 3, and is called *triple* ratio; in $24:6$ the ratio is 4, and is called *quadruple* ratio, &c.

We may next consider those relations whose ratios are expressed by fractions, as $12:9$, where the ratio is $\frac{4}{3}$ or $1\frac{1}{3}$; $18:27$, where the ratio is $\frac{2}{3}$, &c. We may also distinguish those relations in which the consequent contains exactly twice, thrice, &c. the antecedent; such are the relations $6:12$, $5:15$, &c. the ratio of which some call, *subduple*, *subtriple*, &c. ratios.

Further, we call that ratio *rational*, which is an expressible number, the antecedent and consequent being integers, as in $11:7$, $8:15$, &c. and we call that an *irrational* or *surd* ratio, which can neither be exactly expressed by integers, nor by fractions, as in $\sqrt{5}:8$, $4:\sqrt{3}$.

389. Let a be the antecedent, b the consequent, and d the ratio, we know already that a and b being given, we find $d = \frac{a}{b}$.

If the consequent b were given with the ratio, we should find the antecedent $a = b\,d$, because $b\,d$ divided by b gives d. Lastly, when the antecedent a is given, and the ratio d, we find the consequent $b = \frac{a}{d}$; for, dividing the antecedent a by the consequent $\frac{a}{d}$, we obtain the quotient d, that is to say, the ratio.

390. Every relation $a:b$ remains the same, though we multiply or divide the antecedent and consequent by the same number, because the ratio is the same. Let d be the ratio of $a:b$, we have $d = \frac{a}{b}$; now the ratio of the relation $n\,a : n\,b$ is also $\frac{a}{b} = d$, and that of the relation $\frac{a}{n} : \frac{b}{n}$ is likewise $\frac{a}{b} = d$.

391. When a ratio has been reduced to its lowest terms, it is easy to perceive and enunciate the relation. For example, when the ratio $\frac{a}{b}$ has been reduced to the fraction $\frac{p}{q}$, we say

$$a:b = p:q,\quad a:b::p:q,$$

which is read, a is to b as p is to q. Thus, the ratio of the relation $6:3$ being $\frac{6}{3}$, or 2, we say $6:3 = 2:1$. We have likewise $18:12 = 3:2$, and $24:18 = 4:3$, and $30:45 = 2:3$, &c. But if the ratio cannot be abridged, the relation will not become more evident; we do not simplify the relation by saying

$$9 : 7 = 9 : 7.$$

392. On the other hand, we may sometimes change the relation of two very great numbers into one that shall be more simple and evident, by reducing both to their lowest terms. For example, we can say $28844 : 14422 = 2 : 1$; or,

$$10566 : 7044 = 3 : 2; \text{ or, } 57600 : 25200 = 16 : 7.$$

393. In order, therefore, to express any relation in the clearest manner, it is necessary to reduce it to the smallest possible numbers. This is easily done, by dividing the two terms of the relation by their greatest common divisor. For example, to reduce the relation $57600 : 25200$ to that of $16 : 7$, we have only to perform the single operation of dividing the numbers 576 and 252 by 36, which is their greatest common divisor.

394. It is important, therefore, to know how to find the greatest common divisor of two given numbers; but this requires a rule, which we shall explain in the following chapter.

CHAPTER VI.

Of the greatest Common Divisor of two given Numbers.

395. There are some numbers which have no other common divisor than unity, and when the numerator and denominator of a fraction are of this nature, it cannot be reduced to a more convenient form. The two numbers 48 and 35, for example, have no common divisor, though each has its own divisors. For this reason we cannot express the relation $48 : 35$ more simply, because the division of two numbers by 1 does not diminish them.

396. But when the two numbers have a common divisor, it is found by the following rule:

Divide the greater of the two numbers by the less; next, divide the preceding divisor by the remainder; what remains in this second division will afterwards become a divisor for a third division, in which the remainder of the preceding division will be the dividend.

We must continue this operation till we arrive at a division that leaves no remainder; the divisor of this division, and consequently the last divisor, will be the greatest common divisor of the two given numbers.

See this operation for the two numbers 576 and 252.

```
252) 576 (2
     504
     ---
      72) 252 (3
          216
          ---
          36) 72 (2
              72
              --
              0.
```

So that, in this instance, the greatest common divisor is 36.

397. It will be proper to illustrate this rule by some other examples. Let the greatest common divisor of the numbers 504 and 312 be required.

```
312) 504 (1
     312
     ---
     192) 312 (1
          192
          ---
          120) 192 (1
               120
               ---
                72) 120 (1
                     72
                    ---
                     48) 72 (1
                         48
                         --
                         24) 48 (2
                             48
                             --
                             0.
```

So that 24 is the greatest common divisor, and consequently the relation 504 : 312 is reduced to the form 21 : 13.

398. Let the relation 625 : 529 be given, and the greatest common divisor of these two numbers be required.

```
529) 625 (1
     529
     ---
      96) 529 (5
          480
          ---
           49) 96 (1
               49
               --
               47) 49 (1
                   47
                   --
                    2) 47 (23
                       46
                       --
                        1) 2 (2
                           2
                           -
                           0.
```

Wherefore, 1 is, in this case, the greatest common divisor, and consequently we cannot express the relation 625 : 529 by less numbers, nor reduce it to less terms.

399. It may be proper, in this place, to give a demonstration of the rule. In order to this, let a be the greater and b the less of the given numbers; and let d be one of their common divisors; it is evident that a and b being divisible by d, we may also divide the quantities $a - b$, $a - 2\,b$, $a - 3\,b$, and, in general, $a - n\,b$ by d.

400. The converse is no less true; that is to say, if the numbers b and $a - n\,b$ are divisible by d, the number a will also be divisible by d. For $n\,b$ being divisible by d, we could not divide $a - n\,b$ by d, if a were not also divisible by d.

401. We observe further, that if d be the *greatest* common divisor of two numbers, b and $a - n\,b$, it will also be the greatest common divisor of the two numbers a and b. Since, if a greater common divisor could be found than d, for these numbers, a and b, that number would also be a common divisor of b and $a - n\,b$; and consequently d would not be the greatest common divisor of these two numbers. Now we have supposed d the greatest divisor common to b and $a - n\,b$; wherefore d must also be the greatest common divisor of a and b.

402. These three things being laid down, let us divide, according to the rule, the greater number a by the less b; and let us suppose the quotient $= n$; the remainder will be $a - n\,b$, which must be less than b. Now this remainder $a - n\,b$ having the same greatest common divisor with b, as the given numbers a and b, we have only to repeat the division, dividing the preceding divisor b by the remainder $a - n\,b$; the new remainder, which we obtain, will still have, with the preceding divisor, the same greatest common divisor, and so on.

403. We proceed in the same manner, till we arrive at a division without a remainder; that is, in which the remainder is nothing. Let p be the last divisor, contained exactly a certain number of times in its dividend; this dividend will therefore be divisible by p, and will have the form $m\,p$; so that the numbers p and $m\,p$, are both divisible by p; and it is certain that they have no greater common divisor, because no number can actually be divided by a number greater than itself. Consequently, this last divisor is also the greatest common divisor of the given numbers a and b, and the rule, which we laid down, is demonstrated.

404. We may give another example of the same rule, requiring the greatest common divisor of the numbers 1728 and 2304. The operation is as follows:

```
1728) 2304 (1
      1728
      ----
       576) 1728 (3
            1728
            ----
              0.
```

From this it follows, that 576 is the greatest common divisor, and that the relation 1728 : 2304 is reduced to 3 : 4, that is to say, 1728 is to 2304 the same as 3 is to 4.

CHAPTER VII.

Of Geometrical Proportions.

405. Two geometrical relations are equal, when their ratios are equal. This equality of two relations is called a *geometrical proportion*; and we write for example $a:b = c:d$, or $a:b::c:d$, to indicate that the relation $a:b$ is equal to the relation $c:d$; but this is more simply expressed by saying a is to b as c to d. The following is such a proportion, $8:4 = 12:6$; for the ratio of the relation $8:4$ is $\frac{2}{1}$, and this is also the ratio of the relation $12:6$.

406. So that $a:b = c:d$ being a geometrical proportion, the ratio must be the same on both sides, and $\frac{a}{b} = \frac{c}{d}$; and reciprocally, if the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal, we have $a:b::c:d$.

407. A geometrical proportion consists therefore of four terms, such, that the first, divided by the second, gives the same quotient as the third divided by the fourth. Hence we deduce an important property, common to all geometrical proportion, which is, that *the product of the first and the last term is always equal to the product of the second and third*; or, more simply, that *the product of the extremes is equal to the product of the means.*

408. In order to demonstrate this property, let us take the geometrical proportion $a:b = c:d$, so that $\frac{a}{b} = \frac{c}{d}$. If we multiply both these fractions by b, we obtain $a = \frac{bc}{d}$, and multiplying both sides further by d, we have $ad = bc$. Now ad is the product of the extreme terms, bc is that of the means, and these two products are found to be equal.

409. *Reciprocally, if the four numbers,* a, b, c, d, *are such that the product of th etwo extremes* a *and* d *is equal to the product of the two means* b *and* c, *we are certain that they form a geometrical proportion.* For since $ad = bc$, we have only to divide both sides by bd, which gives us $\frac{ad}{bd} = \frac{bc}{bd}$ or $\frac{a}{b} = \frac{c}{d}$, and consequently

$$a:b = c:d.$$

410. *The four terms of a geometrical proportion, as* a : b = c d, *may be transposed in different ways, without destroying the proportion. For the rule being always, that the product of the extremes is equal to the product of the means, or* a d = b c, *we may say:*

1st. b : a = d : c; 2dly. a : c = b : d; 3dly. d : b = c : a; 4thly. d : c = b : a.

411. Besides these four geometrical proportions, we may deduce some others from the same proportion, $a : b = c : d$. *We may say, the first term, plus the second, is to the first as the third + the fourth is to the third;* that is, $a + b : a = c + d : c$.

We may further say; *the first — the second is to the first, as the third — the fourth is to the third,* or $a - b : a = c - d : c$.

For, if we take the product of the extremes and means, we have $a\,c - b\,c = a\,c - a\,d$, which evidently leads to the equality $a\,d = b\,c$.

Lastly, it is easy to demonstrate, that $a + b : b = c + d : d$; and that $a - b : b = c - d : d$.

412. All the proportions which we have deduced from $a : b = c : d$, may be represented, generally, as follows:

$$m\,a + n\,b : p\,a + q\,b = m\,c + n\,d : p\,c + q\,d.$$

For the product of the extreme terms is

$$m\,p\,a\,c + n\,p\,b\,c + m\,q\,a\,d + n\,q\,b\,d;$$

which, since $a\,d = b\,c$, becomes

$$m\,p\,a\,c + n\,p\,b\,c + m\,q\,b\,c + n\,q\,b\,d.$$

Further, the product of the mean terms is

$$m\,p\,a\,c + m\,q\,b\,c + n\,p\,a\,d + n\,q\,b\,d;$$

or, since $a\,d = b\,c$, it is $m\,p\,a\,c + m\,q\,b\,c + n\,p\,b\,c + n\,q\,b\,d$; so that the two products are equal.

413. It is evident, therefore, that a geometrical proportion being given, for example, $6 : 3 = 10 : 5$, an infinite number of others may be deduced from it. We shall give only a few:

$$3 : 6 = 5 : 10;\ 6 : 10 = 3 : 5;\ 9 : 6 = 15 : 10;$$
$$3 : 3 = 5 : 5;\ 9 : 15 = 3 : 5;\ 9 : 3 = 15 : 5.$$

414. Since, in every geometrical proportion, the product of the extremes is equal to the product of the means, we may, when the three first terms are known, find the fourth from them. Let the

three first terms be $24 : 15 = 40$ to .... as the product of the means is here 600, the fourth term multiplied by the first, that is, by 24, must also make 600; consequently, by dividing 600 by 24, the quotient 25 will be the fourth term required, and the whole proportion will be $24 : 15 = 40 : 25$. In general, therefore, if the three first terms are $a : b = c : \ldots.$ we put d for the unknown fourth letter; and since $a\,d = b\,c$, we divide both sides by a and have $d = \frac{b\,c}{a}$. So that the *fourth term* is $= \frac{b\,c}{a}$, and *is found by multiplying the second term by the third, and dividing that product by the first term.*

415. This is the foundation of the celebrated *Rule of Three* in arithmetic; for what is required in that rule? We suppose three numbers given, and seek a fourth, which may be in geometrical proportion; so that the first may be to the second, as the third is to the fourth.

416. Some particular circumstances deserve attention here.

First, *if in two proportions the first and the third terms are the same,* as in $a : b = c : d$, and $a : f = c : g$, I say that *the two second and the two fourth terms will also be in geometrical proportion,* and that $b : d = f : g$. For, the first proportion being transformed into this, $a : c = b : d$, and the second into this, $a : c = f : g$, it follows that the relations $b : d$ and $f : g$ are equal, since each of them is equal to the relation $a : c$. For example, if $5 : 100 = 2 : 40$, and $5 : 15 = 2 : 6$, we must have $100 : 40 = 15 : 6$.

417. But if the two proportions are such, that the mean terms are the same in both, I say that the first terms will be in an inverse proportion to the fourth terms. That is to say, if $a : b \quad c : d$, and $f : b = c : g$, it follows that $a : f = g : d$. Let the proportions be, for example, $24 : 8 = 9 : 3$, and $6 : 8 = 9 : 12$, we have $24 : 6 = 12 : 3$. The reason is evident; the first proportion gives $a\,d = b\,c$; the second gives $f\,g = b\,c$; therefore,

$$a\,d = f\,g, \text{ and } a : f = g : d, \text{ or } a : g :: f : d.$$

418. Two proportions being given, we may always produce a new one, by separately multiplying the first term of the one by the first term of the other, the second by the second, and so on, with respect to the other terms. Thus, the proportions $a : b = c : d$ and $e : f = g : h$ will furnish this, $a\,e : b\,f = c\,g : d\,h$. For the first giving $a\,d = b\,c$, and the second giving $e\,h = f\,g$, we have also

$$a\,d\,e\,h = b\,c\,f\,g.$$

Now $a\,d\,e\,h$ is the product of the extremes, and $b\,c\,f\,g$ is the product of the means in the new proportion; so that the two products being equal, the proportion is true.

419. Let the two proportions be, for example, $6:4 = 15:10$ and $9:12 = 15:20$, their combination will give the proportion

$$6 \times 9 : 4 \times 12 = 15 \times 15 : 10 \times 20,$$
$$\text{or } 54:48 = 225:200,$$
$$\text{or } 9:8 = 9:8.$$

420. We shall observe lastly, that if two products are equal, $a\,d = b\,c$, we may reciprocally convert this equality into a geometrical proportion; for we shall always have one of the factors of the first product, in the same proportion to one of the factors of the second product, as the other factor of the second product is to the other factor of the first product; that is, in the present case, $a:c = b:d$, or $a:b = c:d$. Let $3 \times 8 = 4 \times 6$, and we may form from it this proportion, $8:4 = 6:3$, or this, $3:4 = 6:8$. Likewise, if $3 \times 5 = 1 \times 15$, we shall have

$$3:15 = 1:5, \text{ or } 5:1 = 15:3, \text{ or } 3:1 = 15:5.$$

CHAPTER VIII.

Observations on the Rules of Proportion and their Utility.

421. This theory is so useful in the occurrences of common life, that scarcely any person can do without it. There is always a proportion between prices and commodities; and when different kinds of money are the subject of exchange, the whole consists in determining their mutual relations. The examples, furnished by these reflections, will be very proper for illustrating the principles of proportion, and showing their utility by the application of them.

422. If we wished to know, for example, the relation between two kinds of money; suppose an old louis d'or and a ducat; we must first know the value of those pieces, when compared to others

of the same kind. Thus, an old louis being, at Berlin, worth 5 rix dollars* and 8 drachms, and a ducat being worth 3 rix dollars, we may reduce these two values to one denomination; either to rix dollars, which gives the proportion 1 L : 1 D = $5\frac{1}{3}$ R 3 : R, or = 16 : 9; or to drachms, in which case we have 1 L : 1 D = 128 : 72 = 16 : 9. These proportions evidently give the true relation of the old louis to the ducat; for the equality of the products of the extremes and the means gives, in both, 9 louis = 16 ducats; and, by means of this comparison we may change any sum of old louis into ducats, and *vice versâ*. Suppose it were required to tell how many ducats there are in 1000 old louis, we have this rule of three. If 9 louis are equal to 16 ducats, what are 1000 louis equal to? The answer will be $1777\frac{7}{9}$ ducats.

If, on the contrary, it were required to find how many old louis d'or there are in 1000 ducats, we have the following proportion. If 16 ducats are equal to 9 louis; what are 1000 ducats equal to? *Answer*, $562\frac{1}{2}$ old louis d'or.

423. Here (at Petersburg), the value of the ducat varies, and depends on the course of exchange. This course determines the value of the ruble in stivers, or Dutch half-pence, 105 of which make a ducat.

So that when the exchange is at 45 stivers, we have this proportion, 1 ruble : 1 ducat = 45 : 105 = 3 : 7; and hence this equality, 7 rubles = 3 ducats.

By this we shall find the value of a ducat in rubles; for 3 ducats : 7 rubles = 1 ducat : ..... *Answer*, $2\frac{1}{3}$ rubles.

If the exchange were at 50 stivers, we should have this proportion, 1 ruble : 1 ducat = 50 : 105 = 10 : 21, which would give 21 rubles = 10 ducats; and we should have 1 ducat = $2\frac{1}{10}$ rubles. Lastly, when the exchange is at 44 stivers, we have 1 ruble : 1 ducat = 44 : 105, and consequently 1 ducat = $2\frac{17}{44}$ rubles = 2 rubles $38\frac{7}{11}$ copecks.†

424. It follows from this, that we may also compare different kinds of money, which we have frequently occasion to do in bills of exchange. Suppose, for example, that a person of this place has

* The rix dollar of Germany is valued at 92 cents 6 mills, and a drachm is one twenty-fourth part of a rix dollar.

† A copeck is $\frac{1}{100}$ part of a ruble, as is easily deduced from the above.

1000 rubles to be paid to him at Berlin, and that he wishes to know the value of this sum in ducats at Berlin.

The exchange is here at $47\frac{1}{2}$, that is to say, one ruble makes $47\frac{1}{2}$ stivers. In Holland, 20 stivers make a florin; $2\frac{1}{2}$ Dutch florins make a Dutch dollar. Further, the exchange of Holland with Berlin is at 142, that is to say, for 100 Dutch dollars, 142 dollars are paid at Berlin. Lastly, the ducat is worth 3 dollars at Berlin.

425. To resolve the questions proposed, let us proceed step by step. Beginning therefore with the stivers, since 1 ruble = $47\frac{1}{2}$ stivers, or 2 rubles = 95 stivers, we shall have 2 rubles : 95 stivers = 1000 : .... *Answer*, 47500 stivers. If we go further and say 20 stivers : 1 florin = 47500 stivers : .... we shall have 2375 florins. Further, $2\frac{1}{2}$ florins = 1 Dutch dollar, or 5 florins = 2 Dutch dollars; we shall therefore have 5 florins : 2 Dutch dollars = 2375 florins : .... *Answer*, 950 Dutch dollars.

Then taking the dollars of Berlin, according to the exchange at 142, we shall have 100 Dutch dollars : 142 dollars = 950 : the fourth term, 1349 dollars of Berlin. Let us, lastly, pass to the ducats, and say 3 dollars : 1 ducat = 1349 dollars : .... *Answer*, $449\frac{2}{3}$ ducats.

426. In order to render these calculations still more complete, let us suppose that the Berlin banker refuses, under some pretext or other, to pay this sum, and to accept the bill of exchange without five per cent. discount; that is, paying only 100 instead of 105. In that case, we must make use of the following proportion; 105 : 100 = $449\frac{2}{3}$: a fourth term, which is $428\frac{16}{63}$ ducats.

427. We have shown that six operations are necessary, in making use of the Rule of Three; but we can greatly abridge those calculations, by a rule, which is called the ***Rule of Reduction.*** To explain this rule, we shall first consider the two antecedents of each of the six operations.

I.	2 rubles	:	95 stivers.
II.	20 stivers	:	1 Dutch flor.
III.	5 Dutch flor.	:	2 Dutch doll.
IV.	100 Dutch doll.	:	142 dollars.
V.	3 dollars	:	1 ducat.
VI.	105 ducats	:	100 ducats.

If we now look over the preceding calculations, we shall observe, that we have always multiplied the given sum by the second terms, and that we have divided the products by the first; it is evident therefore, that we shall arrive at the same results, by multiplying, at once, the sum proposed by the product of all the second terms, and dividing by the product of all the first terms. Or, which amounts to the same thing, that we have only to make the following proportion; as the product of all the first terms is to the product of all the second terms, so is the given number of rubles to the number of ducats payable at Berlin.

428. This calculation is abridged still more, when amongst the first terms are found some that have common divisors with some of the second terms; for, in this case, we destroy those terms, and substitute the quotient arising from the division by that common divisor. The preceding example will, in this manner, assume the following form.*

Rubles ~~2~~.	:	19,~~95~~ stiv. 1000 rubles.
~~20~~.	:	1 Dutch flor.
~~5~~.	:	~~2~~ Dutch dollars.
100.	:	142 dollars.
3.	:	1 ducat.
~~105~~,21.	:	~~5~~,~~100~~ ducats.
63~~00~~	:	2698 = 10~~00~~ : —

7) 26980.

9) 3854 (2

428 (2. *Answer*, $428\frac{16}{63}$ ducats.

429. The method which must be observed, in using the rule of reduction, is this; we begin with the kind of money in question, and compare it with another, which is to begin the next relation, in which we compare this second kind with a third, and so on. Each relation, therefore begins with the same kind, as the preceding relation ended with. This operation is continued, till we arrive at the kind of money which the answer requires, and, at the end, we reckon the fractional remainders.

* Divide the 1st and 9th by 2, the 3d and 12th by 20, the 5th and 12th (which is now 5) by 5, also the 2d and 11th by 5.

430. Other examples are added to facilitate the practice of this calculation.

If ducats gain at Hamburg 1 per cent. on two dollars banco; that is to say, if 50 ducats are worth, not 100, but 101 dollars banco; and if the exchange between Hamburg and Konigsberg, is 119 drachms of Poland; that is, if 1 dollar banco gives 119 Polish drachms, how many Polish florins will 1000 ducats give?

30 Polish drachms make 1 Polish florin.

Ducat 1	:	$\not{2}$ doll. B°.	1000 duc.
$\not{1}\not{0}\not{0}$,50	:	101 doll. B°.	
1	:	119 Pol. dr.	
30	:	1 Pol. flor.	

15$\not{0}\not{0}$: 12019 = 10$\not{0}\not{0}$ duc. : . . . .

3) 120190

5) 40063 (1

8012 (3. *Answer*, $8012\frac{2}{3}$ P. fl.

431. We may abridge a little further, by writing the number, which forms the third term, above the second row; for then the product of the second row, divided by the product of the first row, will give the answer sought.

Question. Ducats of Amsterdam are brought to Leipsic, having in the former city the value of 5 flor. 4 stivers current; that is to say, 1 ducat is worth 104 stivers, and 5 ducats are worth 26 Dutch florins. If, therefore, the *agio of the bank** at Amsterdam is 5 per cent. that is, if 105 currency are equal to 100 banco, and if the exchange from Leipsic to Amsterdam, in bank money, is $33\frac{1}{3}$ per cent. that is, if for 100 dollars we pay at Leipsic $133\frac{1}{3}$ dollars; lastly, 2 Dutch dollars making 5 Dutch florins; it is required to find how many dollars we must pay at Leipsic, according to these exchanges, for 1000 ducats?

* The difference of value between bank money and current money

$\not{5}$, $\not{1}\not{0}\not{0}0$ ducats.

Ducats	$\not{5}$	:	26 flor. Dutch curr.
	$\not{1}\not{0}\not{5}$, 21	:	4, $\not{2}\not{0}$, $\not{1}\not{0}0$ flor. Dutch banco.
	$\not{4}\not{0}0$, $\not{2}$	:	533 doll. of Leipsic.
	$\not{5}$	:	$\not{2}$ doll. banco.
	21	:	3) 55432 (1.
			7) 18477 (4.
			2639.

Answer, $2639\frac{13}{21}$ dollars, or 2639 dollars and 15 drachms.

CHAPTER IX.

Of Compound Relations.

432. COMPOUND RELATIONS are obtained, by multiplying the terms of two or more relations, the antecedents by the antecedents, and the consequents by the consequents; we say then, that the relation between those two products is *compounded* of the relations given.

Thus, the relations $a:b$, $c:d$, $e:f$, give the compound relation $ace:bdf$.*

433. A relation continuing always the same, when we divide both its terms by the same number, in order to abridge it, we may greatly facilitate the above composition by comparing the antecedents and the consequents, for the purpose of making such reductions as we performed in the last chapter.

For example, we find the compound relation of the following given relations, thus;

* Each of these three ratios is said to be one of the *roots* of the compound ratio.

Relations given.

12 : 25, 28 : 33, and 55 : 56.

$$\begin{array}{lcl} \cancel{12}, \cancel{4}, 2 & : & 5, \cancel{25}. \\ \cancel{28} & : & \cancel{11}, \cancel{33}. \\ \cancel{55}, \cancel{11} & : & \cancel{2}, \cancel{56}. \\ \hline 2 & : & 5. \end{array}$$

So that 2 : 5 is the compound relation required.

434. The same operation is to be performed, when it is required to calculate generally by letters; and the most remarkable case is that, in which each antecedent is equal to the consequent of the preceding relation. If the given relations are

$$\begin{array}{c} a : b \\ b : c \\ c : d \\ d : e \\ e : a \end{array}$$

the compound relation is 1 : 1.

435. The utility of these principles will be perceived, when it is observed, that the relation between two square fields is compounded of the relations of the lengths and the breadths.

Let the two fields, for example, be A and B; let A have 500 feet in length by 60 feet in breadth, and let the length of B be 360 feet, and its breadth 100 feet; the relation of the lengths will be 500 : 360, and that of the breadths 60 : 100. So that we have

$$\begin{array}{lcl} \cancel{500}, 5 & : & 6, \cancel{360}. \\ \cancel{60} & : & \cancel{100}. \\ \hline 5 & : & 6 \end{array}$$

Wherefore the field A is to the field B, as 5 to 6.

436. *Another Example.* Let the field A be 721 feet long, 88 feet broad; and let the field B be 660 feet long, and 90 feet broad; the relations will be compounded in the following manner.

$$\begin{array}{llcl} \text{Relation of the lengths,} & \cancel{720}, 8 & : & 15, \cancel{60}, \cancel{660} \\ \text{Relation of the breadths,} & \cancel{88}, \cancel{8}, 2 & : & \cancel{90} \\ \hline \text{Relation of the fields A and B,} & 16 & : & 15. \end{array}$$

437. Further, if it be required to compare two chambers with respect to the space, or contents, we observe that that relation is compounded of three relations; namely, of that of the lengths, that of the breadths, and that of the heights. Let there be, for example, the chamber A, whose length = 36 feet, breadth = 16 feet, and height = 14 feet, and the chamber B, whose length = 42 feet, breadth = 24 feet, and height = 10 feet; we shall have these three relations;

For the length	~~36~~, ~~6~~	:	~~7~~, ~~42~~.
For the breadth	~~16~~, ~~4~~, 2	:	~~6~~, ~~24~~.
For the height	~~14~~, 2	:	5, ~~10~~.
	4	:	5

So that the contents of the chamber A : contents of the chamber B, as 4 : 5.

438. When the relations which we compound in this manner are equal, there result multiplicate relations. Namely, two equal relations give a ***duplicate ratio*** or ***ratio of the squares***; three equal relations produce the ***triplicate ratio*** or ***ratio of the cubes***, and so on; for example, the relations $a : b$ and $a : b$ give the compound relation $a\,a : b\,b$; wherefore we say, that the squares are in the duplicate ratio of their roots. And the ratio $a : b$ multiplied thrice, giving the ratio $a^3 : b^3$, we say that the cubes are in the triplicate ratio of their roots.

439. Geometry teaches, that two circular spaces are in the duplicate relation of their diameters; this means, that they are to each other as the squares of their diameters.

Let A be a circular space having the diameter = 45 feet, and B another circular space, whose diameter = 30 feet; the first space will be to the second, as 45 × 45 to 30 × 30; or, compounding these two equal relations,

~~45~~, ~~9~~, 3	:	2, ~~6~~, ~~30~~.
~~45~~, ~~9~~, 3	:	2, ~~6~~, ~~30~~.
9	:	4.

Wherefore the two areas are to each other as 9 to 4.

440. It is also demonstrated, that the solid contents of spheres are in the ratio of the cubes of the diameters. Thus, the diameter of a

globe A, being 1 foot, and the diameter of a globe B, being 2 feet, the solid contents of A will be those of B, as $1^3 : 2^3$; or, as 1 to 8.

If, therefore, the spheres are formed of the same substance, the sphere B will weigh 8 times as much as the sphere A.

441. It is evident, that we may, in this manner, find the weight of cannon balls, their diameters and the weight of one, being given. For example, let there be the ball A, whose diameter = 2 inches, and weight = 5 pounds; and, if the weight of another ball be required, whose diameter is 8 inches, we have this proportion, $2^3 : 8^3 = 5$ to the fourth term, 320 pounds, which gives the weight of the ball B. For another ball C, whose diameter = 15 inches, we should have,

$$2^3 : 15^3 = 5 : \ldots \ \textit{Answer}, \ 2109\tfrac{3}{8} \text{ lb.}$$

442. When the ratio of two fractions, as $\frac{a}{b} : \frac{c}{d}$, is required, we may always express it in integer numbers; for we have only to multiply the fractions by $b\,d$, in order to obtain the ratio $a\,d : b\,c$, which is equal to the other; from which results the proportion

$$\frac{a}{b} : \frac{c}{d} = a\,d : b\,c.$$

If, therefore, $a\,d$ and $b\,c$ have common divisors, the ratio may be reduced to less terms. Thus,

$$\tfrac{15}{24} : \tfrac{25}{36} = 15 \times 36 : 24 \times 25 = 9 : 10.$$

443. If we wished to know the ratio of the fractions $\frac{1}{a}$ and $\frac{1}{b}$, it is evident that we should have $\frac{1}{a} : \frac{1}{b} = b : a$; which is expressed by saying, that *two fractions, which have unity for their numerator, are in the reciprocal, or inverse ratio of their denominators. The same may be said of two fractions, which have any common numerator*; for $\frac{c}{a} : \frac{c}{b} = b : a$. *But if two fractions have their denominators equal, as* $\frac{a}{c} : \frac{b}{c}$, *they are in the direct ratio of the numerators;* namely, as $a : b$. Thus, $\frac{3}{8} : \frac{3}{16} = \frac{6}{16} : \frac{3}{16} = 6 : 3 = 2 : 1$, and

$$\tfrac{10}{7} : \tfrac{15}{7} = 10 : 15, \text{ or, } = 2 : 3.$$

444. It is observed, that in the free descent of bodies, a body falls 16* feet in a second, that in two seconds of time it falls 64 feet, and that in three seconds it falls 144 feet; hence it is concluded, that the heights are to one another as the squares of the times; and that, reciprocally, the times are in the subduplicate ratio of the heights, or as the square roots of the heights.

If, therefore, it be required to find how long a stone must take to fall from the height of 2304 feet; we have 16 : 2304 = 1 to the square of the time sought. So that the square of the time sought is 144; and, consequently, the time required is 12 seconds.

445. It is required to find how far, or through what height, a stone will pass, by descending for the space of an hour; that is, 3600 seconds. We say, therefore, as the squares of the times, that is, $1^2 : 3600^2$; so is the given height = 16 feet, to the height required.

1 : 12960000 = 16 : .... 207360000 height required.

```
     16
 ---------
  77760000
  1296
 ---------
 207360000
```

If we now reckon 19200 feet for a league, we shall find this height to be 10800; and consequently, nearly four times greater than the diameter of the earth.

446. It is the same with regard to the price of precious stones, which are not sold in the proportion of their weight; every body knows that their prices follow a much greater ratio. The rule for diamonds is, that the price is in the duplicate ratio of the weight, that is to say, the ratio of the prices is equal to the square of the ratio of the weights. The weight of diamonds is expressed in carats, and a carat is equivalent to 4 grains; if, therefore, a diamond of one carat is worth 10 livres, a diamond of 100 carats will be worth as many times 10 livres, as the square of 100 contains 1; so that we shall have, according to the rule of three,

$1^2 : 100^2 = 10$ livres,

or 1 : 10000 = 10 : .... *Answer*, 100000 livres.

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* 15 is used in the original, as expressing the descent in Paris feet. It is here altered to English feet.

There is a diamond in Portugal, which weighs 1680 carats; its price will be found, therefore, by making

$$1^2 : \quad 1680^2 = 10 \text{ liv.} : \ldots \text{ or}$$
$$1 : 2822400 = 10 : 28224000 \text{ liv.}$$

447. The posts, or mode of travelling, in France furnish examples of compound ratios, as the price is according to the compound ratio of the number of horses, and the number of leagues or posts. For example, one horse costing 20 sous per post, it is required to find how much is to be paid for 28 horses and $4\frac{1}{2}$ posts.

We write the first ratio of horses, 1 : 28,
Under this ratio we put that of the stages or posts, 2 : 9,

And, compounding the two ratios, we have 2 : 252,
Or, 1 : 126 = 1 livre to 126 francs, or 42 crowns.

Another Question. If I pay a ducat for 8 horses, for 3 German miles, how much must I pay for thirty horses for four miles? The calculation is as follows:

8, 4 : 5, 15, 30,
3 : 4,

1 : 5, = 1 ducat : the 4th term, which will be 5 ducats.

448. The same composition occurs, when workmen are to be paid, since those payments generally follow the ratio compounded of the number of workmen, and that of the days which they have been employed.

If, for example, 25 sous per day be given to one mason, and it is required to find what must be paid to 24 masons who have worked for 50 days; we state this calculation;

1 : 24
1 : 50

1 : 1200 = 25 : .... 1500 francs.
25

20) 30000 (1500.

As, in such examples, five things are given, the rule, which serves to resolve them, is sometimes called, in books of arithmetic, **The Rule of Five.**

CHAPTER X.

Of Geometrical Progressions.

449. A SERIES of numbers, which are always becoming a certain number of times greater or less, is called a *geometrical progression,* because each term is constantly to the following one in the same geometrical ratio. And the number which expresses how many times each term is greater than the preceding, is called the *exponent.* Thus, when the first term is 1 and the exponent = 2, the geometrical progression becomes,

Terms	1	2	3	4	5	6	7	8	9 &c.
Prog.	1,	2,	4,	8,	16,	32,	64,	128,	256, &c.

the numbers 1, 2, 3, &c. always marking the place which each term holds in the progression.

450. If we suppose, in general, the first term = a, and the exponent = b, we have the following geometrical progression;

	1,	2,	3,	4,	5,	6,	7,	8	.... n
Prog.	a,	ab,	ab^2,	ab^3,	ab^4,	ab^5,	ab^6,	ab^7	.... ab^{n-1}.

So that, when this progression consists of n terms, the last term is $= a\,b^{n-1}$. We must remark here, that if the exponent b be greater than unity, the terms increase continually; if the exponent $b = 1$, the terms are all equal; lastly, if the exponent b be less than 1, or a fraction, the terms continually decrease. Thus, when $a = 1$ and $b = \frac{1}{2}$, we have this geometrical progression;

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{64}, \tfrac{1}{128}, \text{\&c.}$$

451. Here therefore we have to consider;

I. The first term, which we have called a.

II. The exponent, which we call b.

III. The number of terms, which we have expressed by n.

IV. The last term, which we have found $= a\,b^{n-1}$.

So that, when the three first of these are given, the last term is found by multiplying the $n - 1$ power of b, or b^{n-1}, by the first term a.

If, therefore, the 50th term of the geometrical progression 1, 2, 4, 8, &c. were required, we should have $a = 1$, $b = 2$, and $n = 50$; consequently the 50th term $= 2^{49}$. Now 2^9 being

$= 512$; 2^{10} will be $= 1024$. Wherefore the square of 2^{10}, or 2^{20}, $= 1048576$, and the square of this number, or $1099511627776 = 2^{40}$. Multiplying therefore this value of 2^{40} by 2^9, or by 512, we have 2^{49} equal to 562949953421312.

452. One of the principal questions, which occurs on this subject, is to find the sum of all the terms of a geometrical progression; we shall therefore explain the method of doing this. Let there be given, first, the following progression, consisting of ten terms;

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512,$$

the sum of which we shall represent by s, so that

$$s = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512;$$

doubling both sides, we shall have

$$2s = 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024.$$

Subtracting from this the progression represented by s, there remains $s = 1024 - 1 = 1023$; wherefore the sum required is 1023.

453. Suppose now, in the same progression, that the number of terms is undetermined and $= n$, so that the sum in question, or s, $= 1 + 2 + 2^2 + 2^3 + 2^4 \ldots 2^{n-1}$. If we multiply by 2, we have $2s = 2 + 2^2 + 2^3 + 2^4 \ldots 2^n$, and subtracting from this equation the preceding one, we have $s = 2^n - 1$. We see, therefore, that the sum required is found, by multiplying the last term 2^{n-1}, by the exponent 2, in order to have 2^n, and subtracting unity from that product.

454. This is made still more evident by the following examples, in which we substitute successively, for n, the numbers 1, 2, 3, 4, &c.

$1 = 1$; $1 + 2 = 3$; $1 + 2 + 4 = 7$; $1 + 2 + 4 + 8 = 15$; $1 + 2 + 4 + 8 + 16 = 31$; $1 + 2 + 4 + 8 + 16 + 32 = 63$, &c.

455. On this subject the following question is generally proposed. A man offers to sell his horse by the nails in his shoes, which are in number 32; he demands 1 liard for the first nail, 2 for the second, 4 for the third, 8 for the fourth, and so on, demanding for each nail twice the price of the preceding. It is required to find what would be the price of the horse?

This question is evidently reduced to finding the sum of all the terms of the geometrical progression, 1, 2, 4, 8, 16, &c. continued to the 32d term. Now this last term is 2^{31}; and, as we have already found $2^{20} = 1048576$, and $2^{10} = 1024$, we shall have $2^{20} \times 2^{10} = 2^{30}$ equal to 1073741824; and multiplying again by 2, the last term $2^{31} = 2147483648$; doubling therefore this number, and subtracting unity from the product, the sum required becomes 4294967295 liards. These liards make 1073741823¾ sous, and dividing by 20, we have 53687091 livres, 8 sous, 9 deniers for the sum required.

456. Let the exponent now be $= 3$, and let it be required to find the sum of the geometrical progression 1, 3, 9, 27, 81, 243, 729, consisting of 7 terms. Suppose it $= s$, so that

$$s = 1 + 3 + 9 + 27 + 81 + 243 + 729;$$

we shall then have, multiplying by 3,

$$3s = 3 + 9 + 27 + 81 + 243 + 729 + 2187;$$

and subtracting the preceding series, we have

$$2s = 2187 - 1 = 2186.$$

So that the double of the sum is 2186, and consequently the sum required $= 1093$.

457. In the same progression, let the number of terms $= n$, and the sum $= s$; so that $s = 1 + 3 + 3^2 + 3^3 + 3^4 + \ldots . 3^{n-1}$. If we multiply by 3, we have $3s = 3 + 3^2 + 3^3 + 3^4 + \ldots . 3^n$. Subtracting from this the value of s, as all the terms of it, except the first, destroy all the terms of the value of $3s$, except the last, we shall have $2s = 3^n - 1$; therefore $s = \frac{3^n - 1}{2}$. So that the sum required is found by multiplying the last term by 3, subtracting 1 from the product, and dividing the remainder by 2. This will appear, also, from the following examples;

$$1 = 1;\ 1 + 3 = \frac{3 \times 3 - 1}{2} = 4;\ 1 + 3 + 9 = \frac{3 \times 9 - 1}{2} = 13;$$

$$1 + 3 + 9 + 27 = \frac{3 \times 27 - 1}{2} = 40;\ 1 + 3 + 9 + 27 + 81 =$$

$$\frac{3 \times 81 - 1}{2} = 121.$$

458. Let us now suppose, generally, the first term $= a$, the exponent $= b$, the number of terms $= n$, and their sum $= s$, so that

$$s = a = ab + ab^2 + ab^3 + ab^4 + \ldots . ab^{n-1}.$$

If we multiply by b, we have

$$bs = ab + ab^2 + ab^3 + ab^4 + ab^5 + \ldots . ab^n,$$

and subtracting the above equation, there remains

$$(b-1)s = ab^n - a;$$

whence we easily deduce the sum required $s = \frac{ab^n - a}{b-1}$. *Consequently the sum of any geometrical progression is found by multiplying the last term by the exponent of the progression, subtracting the first term from the product, and dividing the remainder by the exponent minus unity.*

459. Let there be a geometrical progression of seven terms, of which the first $= 3$; and let the exponent be $= 2$; we shall then have $a = 3$, $b = 2$, and $n = 7$; wherefore the last term $= \times 2^6$, or $3 \times 64 = 192$; and the whole progression will be

3, 6, 12, 24, 48, 96, 192.

Further, if we multiply the last term 192 by the exponent 2, we have 384; subtracting the first term, there remains 381; and dividing this by $b - 1$, or by 1, we have 381 for the sum of the whole progression.

460. Again, let there be a geometrical progression of six terms; let 4 be the first, and let the exponent be $= \frac{3}{2}$. The progression is

$$4, 6, 9, \tfrac{27}{2}, \tfrac{81}{4}, \tfrac{243}{8}.$$

If we multiply this last term $\frac{243}{8}$ by the exponent $\frac{3}{2}$, we shall have $\frac{729}{16}$; the subtraction of the first term 4 leaves the remainder $\frac{665}{16}$, which, divided by $b - 1 = \frac{1}{2}$, gives $\frac{665}{8} = 83\frac{1}{8}$.

461. When the exponent is less than 1, and consequently, when the terms of the progression continually diminish, the sum of such a decreasing progression, which would go on to infinity, may be accurately expressed.

For example, let the first term $= 1$, the exponent $= \frac{1}{2}$, and the sum $= s$, so that

$$s = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \&c.$$

ad infinitum.

If we multiply by 2, we have

$$2s = 2 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \&c.$$

ad infinitum.

And, subtracting the preceding progression, there remains $s = 2$ for the sum of the proposed infinite progression.

462. If the first term $= 1$, the exponent $= \frac{1}{3}$, and the sum $= s$; so that

$$s = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \text{\&c. ad infinitum.}$$

Multiplying the whole by 3, we have

$$3s = 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \text{\&c. ad infinitum;}$$

and subtracting the value of s, there remains $2s = 3$; wherefore the sum $s = 1\frac{1}{2}$.

463. Let there be a progression whose sum $= s$, first term $= 2$, and exponent $= \frac{3}{4}$; so that $s = 2 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \frac{81}{128} +$ &c. ad infinitum.

Multiplying by $\frac{4}{3}$, we have $\frac{4}{3}s = \frac{8}{3} + 2 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \frac{81}{128} +$ &c. ad infinitum. Subtracting now the progression s, there remains $\frac{1}{3}s = \frac{8}{3}$; wherefore the sum required $= 8$.

464. If we suppose, in general, the first term $= a$, and the exponent of the progression $= \frac{b}{c}$, so that this fraction may be less than 1, and consequently c greater than b; the sum of the progression, carried on, ad infinitum, will be found thus;

Make $s = a + \frac{ab}{c} + \frac{ab^2}{cc} + \frac{ab^3}{c^3} + \frac{ab^4}{c^4} +$ &c.

Multiplying by $\frac{b}{c}$, we shall have

$$\frac{b}{c}s = \frac{ab}{c} + \frac{ab^2}{c^2} + \frac{ab^3}{c^3} + \frac{ab^4}{c^4} + \text{\&c. ad infinitum.}$$

And, subtracting this equation from the preceding, there remains

$$(1 - \frac{b}{c})\, s = a.$$

Consequently

$$s = \frac{a}{1 - \frac{b}{c}}.$$

If we multiply both terms of this fraction by c, we have

$$s = \frac{ac}{c - b}.$$

The sum of the infinite geometrical progression proposed is, therefore found, by dividing the first term a by 1 minus the exponent, or by multiplying the first term a by the denominator of the exponent, and dividing the product by the same denominator diminished by the numerator of the exponent.

465. In the same manner, we find the sums of progressions, the terms of which are alternately affected by the signs + and —. Let for example,

$$s = a - \frac{ab}{c} + \frac{ab^2}{c^2} - \frac{ab^3}{c^3} + \frac{ab^4}{c^4} - \&c.$$

Multiplying by $\frac{b}{c}$, we have

$$\frac{b}{c}s = \frac{ab}{c} - \frac{ab^2}{c^2} + \frac{ab^3}{c^3} - \frac{ab^4}{c^4} \&c.$$

And, adding this equation to the preceding, we obtain

$$(1 + \frac{b}{c})\, s = a.$$

Whence we deduce the sum required,

$$s = \frac{a}{1 + \frac{b}{c}}, \text{ or } s = \frac{ac}{c + b}.$$

466. We see, then, that if the first term $a = \frac{3}{5}$, and the exponent $= \frac{2}{5}$, that is to say, $b = 2$ and $c = 5$, we shall find the sum of the progression $\frac{3}{5} + \frac{6}{25} + \frac{12}{125} + \frac{24}{625} + \&c. = 1$; since, by subtracting the exponent from 1, there remains $\frac{3}{5}$, and by dividing the first term by that remainder, the quotient is 1.

Further, it is evident, if the terms be alternately positive and negative, and the progression assume this form;

$$\frac{3}{5} - \frac{6}{25} + \frac{12}{125} - \frac{24}{625} + \&c.$$

the sum will be

$$\frac{a}{1 + \frac{b}{c}} = \frac{\frac{3}{5}}{\frac{7}{5}} = \frac{3}{7}.$$

467. *Another Example.* Let there be proposed the infinite progression,

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} + \&c.$$

The first term is here $\frac{3}{10}$, and the exponent is $\frac{1}{10}$. Subtracting this last from 1, there remains $\frac{9}{10}$, and if we divide the first term by this fraction, we have $\frac{1}{3}$ for the sum of the given progression. So that taking only one term of the progression, namely, $\frac{3}{10}$, the error would be $\frac{1}{30}$.

Taking two terms $\frac{3}{10} + \frac{3}{100} = \frac{33}{100}$, there would still be wanting $\frac{1}{300}$ to make the sum $= \frac{1}{3}$.

468. *Another Example.* Let there be given the infinite progression,

$$9 + \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \&c.$$

The first term is 9, the exponent is $\frac{1}{10}$. So that 1, minus the exponent, $= \frac{9}{10}$; and $\frac{\frac{9}{1}}{\frac{9}{10}} = 10$, the sum required.

This series is expressed by a decimal fraction, thus 9,9999999, &c.

CHAPTER XI.

Of Infinite Decimal Fractions.

469. It will be very necessary to show how a vulgar fraction may be transformed into a decimal fraction; and, conversely, how we may express the value of a decimal fraction by a vulgar fraction.

470. *Let it be required, in general, to change the fraction* $\frac{a}{b}$ *into a decimal fraction; as this fraction expresses the quotient of the division of the numerator* a *by the denominator* b, *let us write, instead of* a, *the quantity* a,0000000, *whose value does not at all differ from that of* a, *since it contains neither tenth parts, nor hundredth parts, &c. Let us now divide this quantity by the number* b, *according to the common rules of division, observing to put the point in the proper place, which separates the decimal and the integers.* This is the whole operation, which we shall illustrate by some examples.

Let there be given first the fraction $\frac{1}{2}$, the division in decimals will assume this form,

$$\frac{2)\ 1,0000000}{0,5000000} = \frac{1}{2}.$$

Hence it appears, that $\frac{1}{2}$ is equal to 0,5000000 or to 0,5; which is sufficiently evident, since this decimal fraction represents $\frac{5}{10}$, which is equivalent to $\frac{1}{2}$.

471. Let $\frac{1}{3}$ be the given fraction, and we have,

$$\frac{3)\ 1{,}0000000}{0{,}3333333}\ \&c. = \frac{1}{3}.$$

This shows that the decimal fraction, whose value is $= \frac{1}{3}$, cannot, strictly, ever be discontinued, and that it goes on ad infinitum repeating always the number 3. And, for this reason, it has been already shown, that the fractions $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000}$ &c. ad infinitum, added together make $\frac{1}{3}$.

The decimal fraction, which expresses the value of $\frac{2}{3}$, is also continued ad infinitum, for we have,

$$\frac{3)\ 2{,}0000000}{0{,}6666666}\ \&c. = \frac{2}{3}.$$

And besides, this is evident from what we have just said, because $\frac{2}{3}$ is the double of $\frac{1}{3}$.

472. If $\frac{1}{4}$ be the fraction proposed, we have

$$\frac{4)\ 1{,}0000000}{0{,}2500000}\ \&c. = \frac{1}{4}.$$

So that $\frac{1}{4}$ is equal to 0,2500000, or to 0,25; and this is evident, since $\frac{2}{10} + \frac{5}{100} = \frac{25}{100} = \frac{1}{4}$.

In like manner, we should have for the fraction $\frac{3}{4}$,

$$\frac{4)\ 3{,}0000000}{0{,}7500000} = \frac{3}{4}.$$

So that $\frac{3}{4} = 0{,}75$; and in fact $\frac{7}{10} + \frac{5}{100} = \frac{75}{100} = \frac{3}{4}$.

The fraction $\frac{5}{4}$ is changed into a decimal fraction, by making

$$\frac{4)\ 5{,}0000000}{1{,}2500000} = \frac{5}{4}.$$

Now $1 + \frac{25}{100} = \frac{5}{4}$.

473. In the same manner, $\frac{1}{5}$ will be found equal to 0,2; $\frac{2}{5} = 0{,}4$; $\frac{3}{5} = 0{,}6$; $\frac{4}{5} = 0{,}8$; $\frac{5}{5} = 1$; $\frac{6}{5} = 1, 2$, &c.

When the denominator is 6, we find $\frac{1}{6} = 0{,}1666666$, &c. which is equal to $0{,}666666 - 0{,}5$. Now $0{,}666666 = \frac{2}{3}$, and $0{,}5 = \frac{1}{2}$, wherefore $0{,}1666666 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

We find, also, $\frac{2}{6} = 0{,}333333$, &c. $= \frac{1}{3}$; but $\frac{3}{6}$ becomes $0{,}5000000 = \frac{1}{2}$. Further, $\frac{5}{6} = 0{,}833333 = 0{,}333333 + 0{,}5$, that is to say, $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

474. When the denominator is 7, the decimal fractions become more complicated. For example, we find $\frac{1}{7} = 0,142857$, however it must be observed, that these six figures are repeated continually. To be convinced, therefore, that this decimal fraction precisely expresses the value of $\frac{1}{7}$, we may transform it into a geometrical progression, whose first term is $= \frac{142857}{1000000}$ and the exponent $= \frac{1}{1000000}$; and, consequently, the sum (art. 464) $= \frac{\frac{142857}{1000000}}{1 - \frac{1}{1000000}}$ (multiplying both terms by 1000000) $= \frac{142857}{999999} = \frac{1}{7}$.

475. We may prove, in a manner still more easy, that the decimal fraction which we have found is exactly $= \frac{1}{7}$; for substituting for its value the letter s, we have

$$\begin{aligned}
s &= 0,142857142857142857, \&c.\\
10\,s &= 1,\ 42857142857142857, \&c.\\
100\,s &= 14,\ 2857142857142857, \&c.\\
1000\,s &= 142,\ 857142857142857, \&c.\\
10000\,s &= 1428,\ 57142857142857, \&c.\\
100000\,s &= 14285,\ 7142857142857, \&c.\\
1000000\,s &= 142857,\ 142857142857, \&c.\\
\text{Subtract } s &= \quad\quad 0,\ 142857142857, \&c.\\
\hline
999999\,s &= 142857.
\end{aligned}$$

And, dividing by 999999, we have $s = \frac{142857}{999999} = \frac{1}{7}$. Wherefore the decimal fraction, which was made $= s$, is $= \frac{1}{7}$.

476. In the same manner $\frac{2}{7}$ may be transformed into a decimal fraction, which will be 0,28571428, &c. and this enables us to find more easily the value of the decimal fraction, which we have supposed $= s$; because 0,28571428, &c. must be the double of it, and consequently $= 2\,s$. For we have seen that

$$\begin{aligned}
100\,s &= 14,28571428571 \ \&c.\\
\text{So that subtracting } \ 2\,s &= \ \ 0,28571428571 \ \&c.\\
\hline
\text{there remains } \ 98\,s &= 14\\
\text{wherefore } \ s &= \tfrac{14}{98} = \tfrac{1}{7}.
\end{aligned}$$

We also find $\frac{3}{7} = 0,42857142857$ &c. which, according to our supposition, must be $= 3\,s$; now we have found that

$$\begin{aligned}
10\,s &= 1,42857142857 \ \&c.\\
\text{So that subtracting } 3\,s &= 0,42857142857 \ \&c.\\
\hline
\text{we have } \ 7\,s &= 1, \text{ wherefore } s = \tfrac{1}{7}.
\end{aligned}$$

477. When a proposed fraction, therefore, has the denominator 7, the decimal fraction is infinite, and 6 figures are continually repeated. The reason is, as it is easy to perceive, that when we continue the division we must return, sooner or later, to a remainder which we have had already. Now, in this division, 6 different numbers only can form the remainder, namely, 1, 2, 3, 4, 5, 6; so that, after the sixth division, at furthest, the same figures must return; but when the denominator is such as to lead to a division without remainder, these cases do not happen.

478. Suppose, now, that 8 is the denominator of the fraction proposed; we shall find the following decimal fractions;

$$\tfrac{1}{8} = 0{,}125;\ \tfrac{2}{8} = 0{,}25;\ \tfrac{3}{8} = 0{,}375;\ \tfrac{4}{8} = 0{,}5;\ \tfrac{5}{8} = 0{,}625;$$
$$\tfrac{6}{8} = 0{,}75;\ \tfrac{7}{8} = 0{,}875;\ \&c.$$

If the denominator be 9, we have $\frac{1}{9} = 0{,}111$ &c.; $\frac{2}{9} = 0{,}222$ &c.; $\frac{3}{9} = 0{,}333$ &c.

If the denominator be 10, we have

$$\tfrac{1}{10} = 0{,}1;\ \tfrac{2}{10} = 0{,}2;\ \tfrac{3}{10} = 0{,}3.$$

This is evident from the nature of the thing, as also that $\frac{1}{100} = 0{,}01$; that $\frac{37}{100} = 0{,}37$; that $\frac{256}{1000}$ 0,256; that $\frac{24}{10000} = 0{,}0024$ &c.

479. If 11 be the denominator of the given fraction, we shall have $\frac{1}{11} = 0{,}0909090$ &c. Now, suppose it were required to find the value of this decimal fraction; let us call it s, we shall have $s = 0{,}090909$, and $10\,s = 00{,}909090$; further $100\,s = 9{,}09090$. If therefore, we subtract from the last the value of s, we shall have $99\,s = 9$, and consequently $s = \frac{9}{99} = \frac{1}{11}$. We shall have, also, $\frac{2}{11} = 0{,}181818$ &c.; $\frac{3}{11} = 0{,}2727027$ &c.; $\frac{6}{11} = 0{,}545454$ &c.

480. There is a great number of decimal fractions, therefore, in which one, two, or more figures constantly recur, and which continue thus to infinity. Such fractions are curious, and we shall show how their values may be easily found.

Let us first suppose, that a single figure is constantly repeated, and let us represent it by a, so that $s = 0{,}aaaaaaa$. We have

$$10\,s = a{,}aaaaaaa$$

and subtracting $$s = 0{,}aaaaaaa$$

we have $$9\,s = a;\ \text{wherefore } s = \frac{a}{9}.$$

When two figures are repeated, as $a\ b$, we have $s = 0,ababab a$. Therefore $100\ s = ab,ababab$; and if we subtract s from it, there remains $99\ s = a\ b$; consequently $s = \frac{a\,b}{94}$.

When three figures, as $a\ b\ c$, are found repeated, we have $s = 0,abcabcabc$; consequently, $1000\ s = abc,abcabc$; and subtract s from it, there remains $999\ s = a\ b\ c$; wherefore $s = \frac{a\,b\,c}{999}$, and so on.

Whenever, therefore, a decimal fraction of this kind occurs, it is easy to find its value. Let there be given, for example, 0,296296, its value will be $\frac{296}{999} = \frac{8}{27}$, dividing both terms by 27.

This fraction ought to give again the decimal fraction proposed; and we may easily be convinced that this is the real result, by dividing 8 by 9, and then that quotient by 3, because $27 = 3 \times 9$. We have

9) 8,0000000

3) 0,8888888

0,2962962, &c.

which is the decimal fraction that was proposed.

481. We shall give a curious example by changing the fraction $\frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$, into a decimal fraction. The operation is as follows:

2) 1,00000000000000

3) 0,50000000000000

4) 0,16666666666666

5) 0,04166666666666

6) 0,00833333333333

7) 0,00138888888888

8) 0,00019841269841

9) 0,00002480158730

10) 0,00000275573192

0,00000027557319

SECTION IV.

OF ALGEBRAIC EQUATIONS, AND OF THE RESOLUTION OF THOSE EQUATIONS.

CHAPTER I.

Of the Solution of Problems in general.

ARTICLE 482. The principal object of Algebra, as well as of all the parts of Mathematics, is to determine the value of quantities which were before unknown. This is obtained by considering attentively the conditions given, which are always expressed in known numbers. For this reason Algebra has been defined, *The science which teaches how to determine unknown quantities by means of known quantities.*

483. The definition which we have now given, agrees with all that has been hitherto laid down. We have always seen the knowledge of certain quantities lead to that of other quantities, which before might have been considered as unknown.

Of this, addition will readily furnish an example. To find the sum of two or more given numbers, we had to seek for an unknown number which should be equal to those known numbers taken together.

In subtraction we sought for a number which should be equal to the difference of two known numbers.

A multitude of other examples are presented by multiplication, division, the involution of powers, and the extraction of roots. The question is always reduced to finding, by means of known quantities, another quantity till then unknown.

484. In the last section also, different questions were resolved, in which it was required to determine a number, that could not be deduced from the knowledge of other given numbers, except under certain conditions.

All those questions were reduced to finding, by the aid of some given numbers, a new number which should have a certain connexion with them; and this connexion was determined by certain conditions, or properties, which were to agree with the quantity sought.

485. *When we have a question to resolve, we represent the number sought by one of the last letters of the alphabet, and then consider in what manner the given conditions can form an equality between two quantities.* This equality, which is represented by a kind of formula, called an *equation*, enables us at last to determine the value of the number sought, and consequently to resolve the question. Sometimes several numbers are sought; but they are found in the same manner by equations.

486. Let us endeavour to explain this further by an example. Suppose the following question, or *problem* was proposed.

Twenty persons, men and women, dine at a tavern; the share of the reckoning for one man is 8 sous,* that for one woman is 7 sous, and the whole reckoning amounts to 7 livres 5 sous; required, the number of men, and also of women?

In order to resolve this question, let us suppose that the number of men is $= x$; and now considering this number as known, we shall proceed in the same manner as if we wished to try whether it corresponded with the conditions of the question. Now, the number of men being $= x$, and the men and women making together twenty persons, it is easy to determine the number of the women, having only to subtract that of the men from 20, that is to say, the number of women $= 20 - x$.

But each man spends 8 sous; wherefore x men spend $8x$ sous.

And, since each woman spends 7 sous, $20 - x$ women must spend $140 - 7x$ sous.

So that adding together $8x$ and $140 - 7x$, we see that the whole 20 persons must spend $140 + x$ sous. Now, we know already how much they have spent; namely, 7 livres 5 sous, or 145 sous; there must be an equality therefore between $140 + x$ and

* A sous is $\frac{1}{20}$ of a livre; a livre $\frac{1}{6}$ of a crown, or 17 cents 6 mills.

145; that is to say, we have the equation $140 + x = 145$, and thence we easily deduce $x = 5$.

So that the company consisted of 5 men and 15 women.

487. *Another question of the same kind.*

Twenty persons, men and women, go to a tavern; the men spend 24 florins, and the women as much; but it is found that each man has spent 1 florin more than each woman. Required, the number of men and the number of women?

Let the number of men $= x$.

That of the women will be $= 20 - x$.

Now these x men having spent 24 florins, the share of each man is $\frac{24}{x}$ florins.

Further, the $20 - x$ women having also spent 24 florins, the share of each woman is $\frac{24}{20 - x}$ florins.

But we know that the share of each woman is one florin less than that of each man; if, therefore, we subtract 1 from the share of a man, we must obtain that of a woman; and consequently

$$\frac{24}{x} - 1 = \frac{24}{20 - x}.$$

This, therefore, is the equation from which we are to deduce the value of x. This value is not found with the same ease as in the preceding question; but we shall soon see that $x = 8$, which value corresponds to the equation; for $\frac{24}{8} - 1 = \frac{24}{12}$ includes the equality $2 = 2$.

488. It is evident how essential it is, in all problems, to consider the circumstances of the question attentively, in order to deduce from it an equation, that shall express by letters the numbers sought or unknown. After that, the whole art consists in resolving those equations, or deriving from them the values of the unknown numbers; and this shall be the subject of the present section.

489. We must remark, in the first place, the diversity which subsists among the questions themselves. In some, we seek only for one unknown quantity; in others, we have to find two, or more; and it is to be observed, with regard to this last case, that in order to determine them all, we must deduce from the circumstances, or the conditions of the problem, as many equations as there are unknown quantities.

490. It must have already been perceived, that an equation consists of two parts separated by the sign of equality, $=$, to show that those two quantities are equal to one another. We are often obliged to perform a great number of transformations on those two parts, in order to deduce from them the value of the unknown quantity; but these transformations must be all founded on the following principles; that *two quantities remain equal, whether we add to them, or subtract from them equal quantities; whether we multiply them, or divide them by the same number; whether we raise them both to the same power, or extract their roots of the same degree.*

491. The equations, which are resolved most easily, are those in which the unknown quantity does not exceed the first power, after the terms of the equation have been properly arranged; and we call them *simple equations*, or *equations of the first degree.* But if, after having reduced and ordered an equation, we find in it the square, or the second power of the unknown quantity, it may be called an *equation of the second degree*, which is more difficult to resolve.

CHAPTER II.

Of the Resolution of Simple Equations, or Equations of the First Degree.

492. When the number sought, or the unknown quantity, is represented by the letter x, and the equation we have obtained is such, that one side contains only that x, and the other simply a known number, as for example, $x = 25$, the value of x is already found. We must always endeavour, therefore, to arrive at such a form, however complicated the equation may be when first formed. We shall give, in the course of this section, the rules which serve to facilitate these reductions.

493. Let us begin with the simplest cases, and suppose, first, that we have arrived at the equation $x + 9 = 16$; we see immediately that $x = 7$. And, in general, if we have found $x + a = b$, where a and b express any known numbers, we have only to subtract a

from both sides, to obtain the equation $x = b - a$, which indicates the value of x.

494. If the equation which we have found is $x - a = b$, we add a to both sides, and obtain the value of $x = b + a$.

We proceed in the same manner, if the equation has this form, $x - a = aa + 1$; for we shall have immediately $x = aa + a + 1$.

In this equation, $x - 8a = 20 - 6a$, we find

$$x = 20 - 6a + 8a, \text{ or } x = 20 + 2a.$$

And in this, $x + 6a = 20 + 3a$, we have $x = 20 + 3a - 6a$, or $x = 20 - 3a$.

495. If the original equation has this form, $x - a + b = c$, we may begin by adding a to both sides, which will give $x + b = c + a$; and then subtracting b from both sides, we shall find $x = c + a - b$. But we might also add $+ a - b$ at once to both sides; by this we obtain immediately $x = c + a - b$.

So in the following examples:

If $x - 2a + 3b = 0$, we have $x = 2a - 3b$.

If $x - 3a + 2b = 25 + a + 2b$, we have $x = 25 + 4a$.

If $x - 9 + 6a = 25 + 2a$, we have $x = 34 - 4a$.

496. When the equation which we have found has the form $ax = b$, we only divide the two sides by a, and we have $x = \frac{b}{a}$.

But if the equation has the form $ax + b - c = d$, we must first make the terms that accompany ax vanish, by adding to both sides $-b + c$; and then dividing the new equation, $ax = d - b + c$, by a, we shall have $x = \frac{d - b + c}{a}$.

We should have found the same value by subtracting $+b - c$ from the given equation; that is, we should have had, in the same form, $ax = d - b + c$, and $x = \frac{d - b + c}{a}$. Hence,

If $2x + 5 = 17$, we have $2x = 12$, and $x = 6$.

If $3x - 8 = 7$, we have $3x = 15$, and $x = 5$.

If $4x - 5 - 3a = 15 + 9a$, we have $4x = 20 + 12a$, and, consequently, $x = 5 + 3a$.

497. When the first equation has the form $\frac{x}{a} = b$, we multiply both sides by a, in order to have $x = ab$.

But if we have $\frac{x}{a} + b - c = d$, we must first make

$$\frac{x}{a} = d - b + c,$$

after which we find $x = (d - b + c)\, a = a\, d - a\, b + c$.

Let $\frac{1}{2}\, x - 3 = 4$, we have $\frac{1}{2}\, x = 7$, and $x = 14$.

Let $\frac{1}{3}\, x - 1 + 2\, a = 3 + a$, we have $\frac{1}{3}\, x = 4 - a$, and $x = 12 - 3\, a$.

Let $\frac{x}{a - 1} - 1 = a$, we have $\frac{x}{a - 1} = a + 1$, and $x = a\, a - 1$.

498. When we have arrived at such an equation as $\frac{a\, x}{b} = c$, we first multiply by b, in order to have $a\, x \quad b\, c$, and then dividing by a, we find $x = \frac{b\, c}{a}$.

If $\frac{a\, x}{b} - c = d$, we begin by giving the equation this form

$$\frac{a\, x}{b} = d + c,$$

after which we obtain the value of $a\, x = b\, d + b\, c$, and that of

$$x = \frac{b\, d + b\, c}{a}.$$

Let us suppose $\frac{2}{3}\, x - 4 = 1$, we shall have $\frac{2}{3}\, x = 5$, and $2x = 15$; wherefore $x = \frac{15}{2}$, or $7\frac{1}{2}$.

If $\frac{3}{4}\, x + \frac{1}{2} = 5$, we have $\frac{3}{4}\, x = 5 - \frac{1}{2} = \frac{9}{2}$; wherefore $3\, x = 18$, and $x = 6$.

499. Let us now consider the case, which may frequently occur, in which two or more terms contain the letter x, either on one side of the equation or on both.

If those terms are all on the same side, as in the equation $x + \frac{1}{2}\, x + 5 = 11$, we have $x + \frac{1}{2}\, x = 6$, or $3\, x = 12$, and lastly, $x = 4$.

Let $x + \frac{1}{2}\, x + \frac{1}{3}\, x = 44$, and let the value of x be required: if we first multiply by 3, we have $4\, x + \frac{3}{2}\, x = 132$; then multiplying by 2, we have $11\, x = 264$; wherefore $x = 24$. We might have proceeded more shortly, beginning with the reduction of the three terms which contain x, to the single term $\frac{11}{6}\, x$; and then dividing the equation $\frac{11}{6}\, x = 44$ by 11, we should have had $\frac{1}{6}\, x = 4$, wherefore $x = 24$.

Let $\frac{2}{3}\, x - \frac{3}{4}\, x + \frac{1}{2}\, x = 1$, we shall have, by reduction, $\frac{5}{12}\, x = 1$, and $x = 2\frac{2}{5}$.

Let, more generally, $ax - bx + cx = d$; this is the same as $(a - b + c)\,x = d$, whence we derive $x = \dfrac{d}{a - b + c}$.

500. When there are terms containing x on both sides of the equation, we begin by making such terms vanish from the side from which it is most easily done; that is to say, in which there are fewest of them.

If we have, for example, the equation $3x + 2 = x + 10$, we must first subtract x from both sides, which gives $2x + 2 = 10$; wherefore $2x = 8$, and $x = 4$.

Let $x + 4 = 20 - x$; it is evident that $2x + 4 = 20$; and consequently $2x = 16$, and $x = 8$.

Let $x + 8 = 32 - 3x$, we shall have $4x + 8 = 32$; then $4x = 24$, and $x = 6$.

Let $15 - x = 20 - 2x$, we shall have $15 + x = 20$, and $x = 5$.

Let $1 + x = 5 - \frac{1}{2}x$, we shall have $1 + \frac{3}{2}x = 5$; after that $\frac{3}{2}x = 4$; $3x = 8$; lastly, $x = \frac{8}{3} = 2\frac{2}{3}$.

If $\frac{1}{2} - \frac{1}{3}x = \frac{1}{3} - \frac{1}{4}x$, we must add $\frac{1}{3}x$, which gives $\frac{1}{2} = \frac{1}{3} + \frac{1}{12}x$; subtracting $\frac{1}{3}$, there remains $\frac{1}{12}x = \frac{1}{6}$, and multiplying by 12, we obtain $x = 2$.

If $1\frac{1}{2} - \frac{2}{3}x = \frac{1}{4} + \frac{1}{2}x$, we add $\frac{2}{3}x$, which gives $1\frac{1}{2} = \frac{1}{4} + \frac{7}{6}x$. Subtracting $\frac{1}{4}$, we have $\frac{7}{6}x = 1\frac{1}{4}$, whence we deduce $x = 1\frac{1}{14} = \frac{15}{14}$, by multiplying by 6, and dividing by 7.

501. If we have an equation, in which the unknown number x is a denominator, we must make the fraction vanish, by multiplying the whole equation by that denominator.

Suppose that we have found $\dfrac{100}{x} - 8 = 12$, we first add 8, and have $\dfrac{100}{x} = 20$; then multiplying by x, we have $100 = 20x$; and dividing by 20, we find $x = 5$.

Let $\dfrac{5x + 3}{x - 1} = 7$.

If we multiply by $x - 1$, we have $5x + 3 = 7x - 7$.

Subtracting $5x$, there remains $3 = 2x - 7$.

Adding 7, we have $2x = 10$. Wherefore $x = 5$.

502. Sometimes, also, radical signs are found in equations of the first degree. For example, a number x below 100 is required, and such, that the square root of $100 - x$ may be equal to 8, or $\sqrt{(100 - x)} = 8$; the square of both sides will be $100 - x = 64$,

and adding x we have $100 = 64 + x$; whence we obtain

$$x = 100 - 64 = 36.$$

Or, since $100 - x = 64$, we might have subtracted 100 from both sides; and we should then have had $-x = -36$; whence, multiplying by -1, $x = 36$.

CHAPTER III.

Of the Solution of Questions relating to the preceding Chapter.

503. *Question* I. To divide 7 into two such parts, that the greater may exceed the less by 3.

Let the greater part $= x$, the less will be $= 7 - x$; so that $x = 7 - x + 3$, or $x = 10 - x$; adding x, we have $2x = 10$; and, dividing by 2, the result is $x = 5$.

Answer. The greater part is therefore 5, and the less is 2.

Question II. It is required to divide a into two parts, so that the greater may exceed the less by b.

Let the greater part $= x$, the other will be $a - x$; so that $x = a - x + b$; adding x, we have $2x = a + b$; and dividing by 2, $x = \frac{a+b}{2}$.

Another Solution. Let the greater part $= x$; and, as it exceeds the less by b, it is evident that the less is smaller than the other by b, and therefore must be $= x - b$. Now these two parts, taken together, ought to make a; so that $2x - b = a$; adding b, we have $2x = a + b$, wherefore $x = \frac{a+b}{2}$, which is the value of the greater part; that of the less will be

$$\frac{a+b}{2} - b, \text{ or } \frac{a+b}{2} - \frac{2b}{2}, \text{ or } \frac{a-b}{2}.$$

504. *Question* III. A father, who has three sons, leaves them 1600 crowns. The will specifies, that the eldest shall have 200 crowns more than the second, and that the second shall have 100 crowns more than the youngest. Required the share of each?

Let the share of the third son $= x$; then, that of the second will be $= x + 100$, and that of the first $= x + 300$. Now these three shares make up together 1600 crowns. We have, therefore,

$$3x + 400 = 1600$$
$$3x = 1200$$
$$\text{and } x = 400.$$

Answer. The share of the youngest is 400 crowns; that of the second is 500 crowns; and that of the eldest is 700 crowns.

505. *Question* IV. A father leaves four sons, and 8600 livres; according to the will, the share of the eldest is to be double that of the second, minus 100 livres; the second is to receive three times as much as the third, minus 200 livres; and the third is to receive four times as much as the fourth, minus 300 livres. Required, the respective portions of these four sons?

Let us call x the portion of the youngest; that of the third son will be $= 4x - 300$; that of the second $= 12x - 1100$, and that of the eldest $= 24x - 2300$. The sum of these four shares must make 8600 livres. We have, therefore, the equation

$$41x - 3700 = 8600, \text{ or } 41x = 12300, \text{ and } x = 300.$$

Answer. The youngest must have 300 livres, the third son 900, the second 2500, and the eldest 4900.

506. *Question* V. A man leaves 11000 crowns to be divided between his widow, two sons, and three daughters. He intends that the mother should receive twice the share of a son, and each son to receive twice as much as a daughter. Required, how much each of them is to receive?

Suppose the share of a daughter $= x$, that of a son is consequently $= 2x$, and that of the widow $= 4x$; the whole inheritance is therefore $3x + 4x + 4x$; so that $11x = 11000$, and $x = 1000$.

Answer. Each daughter receives	1000 crowns,
So that the three receive in all	3000
Each son receives 2000 crowns,	
So that both the sons receive	4000
And the mother receives	4000
Sum	11000 crowns.

507. *Question* VI. A father intends, by his will, that his three sons should share his property in the following manner; the eldest

is to receive 1000 crowns less than half the whole fortune; the second is to receive 800 crowns less than the third of the whole property; and the third is to have 600 crowns less than the fourth of the property. Required, the sum of the whole fortune, and the portion of each son?

Let us express the fortune by x.

The share of the first son is	$\frac{1}{2}x - 1000$
That of the second	$\frac{1}{3}x - 800$
That of the third	$\frac{1}{4}x - 600$.

So that the three sons receive in all $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x - 2400$, and this sum must be equal to x.

We have, therefore, the equation $\frac{13}{12}x - 2400 = x$.

Subtracting x, there remains, $\frac{1}{12}x - 2400 = 0$.

Adding 2400, we have $\frac{1}{12}x = 2400$. Lastly, multiplying by 12, the product is x equal to 28800.

Answer. The fortune consists of 28800 crowns, and

The eldest of the sons receives	13400 crowns
The second	8800
The youngest	6600
	28800 crowns.

508. *Question* VII. A father leaves four sons, who share his property in the following manner:

The first takes the half of the fortune, minus 3000 livres.

The second takes the third, minus 1000 livres.

The third takes exactly the fourth of the property.

The fourth takes 600 livres, and the fifth part of the property.

What was the whole fortune, and how much did each son receive?

Let the whole fortune be $= x$;

The eldest of the sons will have	$\frac{1}{2}x - 3000$
The second	$\frac{1}{3}x - 1000$
The third	$\frac{1}{4}x$
The youngest	$\frac{1}{5}x + 600$.

The four will have received in all $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x + \frac{1}{5}x - 3400$, which must be equal to x.

Whence results the equation $\frac{77}{60}x - 3400 = x$;

Subtracting x, we have $\frac{17}{60}x - 3400 = 0$;

Adding 3400, we have $\frac{17}{60}x = 3400$;

Dividing by 17, we have $\frac{1}{20}x = 200$;
Multiplying by 60, we have $x = 12000$.
Answer. The fortune consisted of 12000 livres.

The first son received	3000
The second	3000
The third	3000
The fourth	3000

509. *Question* VIII. To find a number such, that if we add to it its half, the sum exceeds 60 by as much as the number itself is less to 65.

Let the number $= x$, then $x + \frac{1}{2}x - 60 = 65 - x$; that is to say, $\frac{3}{2}x - 60 = 65 - x$;
Adding x, we have $\frac{5}{2}x - 60 = 65$;
Adding 60, we have $\frac{5}{2}x = 125$;
Dividing by 5, we have $\frac{1}{2}x = 25$;
Multiplying by 2, we have $x = 50$.
Answer. The number sought is 50.

510. *Question* IX. To divide 32 into two such parts, that if the less be divided by 6, and the greater by 5, the two quotients taken together may make 6.

Let the less of the two parts sought $= x$; the greater will be $= 32 - x$; the first, divided by 6, gives $\frac{x}{6}$; the second, divided by 5, gives $\frac{32-x}{5}$; now, $\frac{x}{6} + \frac{32-x}{5} = 6$. So that multiplying by 5, we have $\frac{5}{6}x + 32 - x = 30$, or $-\frac{1}{6}x + 32 = 30$.
Adding $\frac{1}{6}x$, we have $32 = 30 + \frac{1}{6}x$.
Subtracting 30, there remains $2 = \frac{1}{6}x$.
Multiplying by 6, we have $x = 12$.
Answer. The two parts are; the less $= 12$, the greater $= 20$.

511. *Question* X. To find such a number, that if multiplied by 5, the product shall be as much less than 40, as the number itself is less than 12.

Let us call this number x. It is less than 12 by $12 - x$. Taking the number x five times, we have $5x$, which is less than 40 by $40 - 5x$, and this quantity must be equal to $12 - x$.
We have therefore $40 - 5x = 12 - x$.
Adding $5x$, we have $40 = 12 + 4x$.
Subtracting 12, we have $28 = 4x$.
Dividing by 4, we have $x = 7$, the number sought.

512. *Question* XI. To divide 25 into two such parts, that the greater may contain the less 49 times.

Let the less part be $= x$, then the greater will be $= 25 - x$. The latter divided by the former ought to give the quotient 49; we have therefore $\frac{25 - x}{x} = 49$.

Multiplying by x, we have $25 - x = 49\,x$.
Adding x $\quad 25 = 50\,x$.
And dividing by 50 $\quad x = \frac{1}{2}$.

Answer. The less of the two numbers sought is $\frac{1}{2}$, and the greater is $24\frac{1}{2}$; dividing therefore the latter by $\frac{1}{2}$, or multiplying by 2, we obtain 49.

513. *Question* XII. To divide 48 into nine parts, so that every part may be always $\frac{1}{2}$ greater than the part which precedes it.

Let the first and least part $= x$, the second will be $= x + \frac{1}{2}$, the third $= x + 1$, &c.

Now these parts form an arithmetical progression, whose first term $= x$; therefore the ninth and last will be $= x + 4$. Adding those two terms together, we have $2\,x + 4$; multiplying this quantity by the number of terms, or by 9, we have $18\,x + 36$; and dividing this product by 2, we obtain the sum of all the nine parts $= 9\,x + 18$; which ought to be equal to 48. We have, therefore, $9\,x + 18 = 48$.

Subtracting 18, there remains $9\,x = 30$.
And dividing by 9, we have $\quad x = 3\frac{1}{3}$.

Answer. The first part is $3\frac{1}{3}$, and the nine parts succeed in the following order:

$$\begin{matrix} 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 \\ 3\frac{1}{3} & + & 3\frac{5}{6} & + & 4\frac{1}{3} & + & 4\frac{5}{6} & + & 5\frac{1}{3} & + & 5\frac{5}{6} & + & 6\frac{1}{3} & + & 6\frac{5}{6} & + & 7\frac{1}{3}, \end{matrix}$$

which together make 48.

514. *Question* XIII. To find an arithmetical progression, whose first term $= 5$, last $= 10$, and sum $= 60$.

Here, we know neither the difference, nor the number of terms; but we know that the first and the last term would enable us to express the sum of the progression, provided only the number of terms was given. We shall, therefore, suppose this number $= x$, and express the sum of the progression by $\frac{15\,x}{2}$; now we know also that this sum is 60; so that $\frac{15\,x}{2} = 60$; $\frac{1}{8}\,x = 4$, and $x = 8$.

Now, since the number of terms is 8, if we suppose the difference $= z$, we have only to seek for the eighth term upon this supposition, and to make it $= 10$. The second term is $5 + z$, the third is $5 + 2z$, and the eighth is $5 + 7z$; so that

$$5 + 7z = 10$$
$$7z = 5$$
$$\text{and } z = \tfrac{5}{7}.$$

Answer. The difference of the progression is $\frac{5}{7}$, and the number of terms is 8; consequently the progression is

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 + & 5\tfrac{5}{7} + & 6\tfrac{3}{7} + & 7\tfrac{1}{7} + & 7\tfrac{6}{7} + & 8\tfrac{4}{7} + & 9\tfrac{2}{7} + & 10 \end{array}$$

the sum of which $= 60$.

515. *Question* XIV. To find such a number that if 1 be subtracted from its double, and the remainder be doubled, then if 2 be subtracted, and the remainder divided by 4, the number resulting from these operations shall be 1 less than the number sought.

Suppose this number $= x$; the double is $2x$; subtracting 1, there remains $2x - 1$; doubling this, we have $4x - 2$; subtracting 2, there remains $4x - 4$; dividing by 4, we have $x - 1$; and this must be one less than x; so that, $x - 1 = x - 1$.

But this is what is called an *identical equation*; and shows that x is indeterminate; or that any number whatever may be substituted for it.

516. *Question* XV. I bought some ells of cloth at the rate of 7 crowns for 5 ells, which I sold again at the rate of 11 crowns for 7 ells, and I gained 100 crowns by the traffic. How much cloth was there?

Suppose that there were x ells of it; we must first see how much the cloth cost. This is found by the following proportion:

If five ells cost 7 crowns; what do x ells cost?

Answer, $\frac{7}{5}x$ crowns.

This was my expenditure. Let us now see my receipt; we must make the following proportion; as 7 ells are to 11 crowns, so are x ells to $\frac{11}{7}x$ crowns.

This receipt ought to exceed the expenditure by 100 crowns; we have, therefore, this equation,

$$\tfrac{11}{7}x = \tfrac{7}{5}x + 100;$$

Subtracting $\frac{7}{5}x$, there remains $\frac{6}{35}x = 100$.

Wherefore $6x = 3500$, and $x = 583\frac{1}{3}$.

Answer. There were $583\frac{1}{3}$ ells, which were bought for $816\frac{2}{3}$ crowns, and sold again for $916\frac{2}{3}$ crowns, by which means the profit was 100 crowns.

517. *Question* XVI. A person buys 12 pieces of cloth for 140 crowns. Two are white, three are black, and seven are blue. A piece of the black cloth costs two crowns more than a piece of the white, and a piece of blue cloth costs three crowns more than a piece of black. Required the price of each kind?

Let a white piece cost x crowns; then the two pieces of this kind will cost $2x$. Further, a black piece costing $x + 2$, the three pieces of this colour will cost $3x + 6$. Lastly, a blue piece costs $x + 5$; wherefore the seven blue pieces cost $7x + 35$. So that the twelve pieces amount in all to $12x + 41$.

Now, the actual and known price of these twelve pieces is 140 crowns; we have, therefore, $12x + 41 = 140$, and $12x = 99$; wherefore $x = 8\frac{1}{4}$;

So that a piece of white cloth costs $8\frac{1}{4}$ crowns; a piece of black cloth costs $10\frac{1}{4}$ crowns, and a piece of blue cloth costs $13\frac{1}{4}$ crowns.

518. *Question* XVII. A man, having bought some nutmegs, says that three nuts cost as much more than one sous as four cost him more than ten liards: Required, the price of those nuts?

We shall call x the excess of the price of three nuts above one sous, or four liards, and shall say; If three nuts cost $x + 4$ liards, four will cost, by the condition of the question, $x + 10$ liards. Now, the price of three nuts gives that of four nuts in another way also, namely, by the rule of three. We make $3 : x + 4 = 4 :$ *Answer,* $\frac{4x + 16}{3}$.

So that $\frac{4x + 16}{3} = x + 10$; or, $4x + 16 = 3x + 30$; wherefore $x + 16 = 30$.
and $x = 14$.

Answer. Three nuts cost 18 liards, and four cost 6 sous; wherefore each cost 6 liards.

519. *Question* XVIII. A certain person has two silver cups, and only one cover for both. The first cup weighs 12 ounces, and if the cover be put on it, it weighs twice as much as the other cup; but if the other cup be covered, it weighs three times as much as the first: Required, the weight of the second cup and that of the cover?

Suppose the weight of the cover $= x$ ounces; the first cup being covered will weigh $x + 12$ ounces. Now this weight being double that of the second cup, this cup must weigh $\frac{1}{2}x + 6$. If it be covered, it will weigh $\frac{3}{2}x + 6$; and this weight ought to be the triple of 12, that is, three times the weight of the first cup. We shall therefore have the equation $\frac{3}{2}x + 6 = 36$, or $\frac{3}{2}x = 30$; wherefore $\frac{1}{2}x = 10$ and $x = 20$.

Answer. The cover weighs 20 ounces, and the second cup weighs 16 ounces.

520. *Question* XIX. A banker has two kinds of change; there must be a pieces of the first to make a crown; and there must be b pieces of the second to make the same sum. A person wishes to have c pieces for a crown; how many pieces of each kind must the banker give him?

Suppose the banker gives x pieces of the first kind; it is evident that he will give $c - x$ pieces of the other kind. Now, the x pieces of the first are worth $\frac{x}{a}$ crown, by the proportion $a : 1 = x : \frac{x}{a}$; and the $c - x$ pieces of the second kind are worth $\frac{c-x}{b}$ crown, because we have $b : 1 = c - x : \frac{c-x}{b}$. So that

$\frac{x}{a} + \frac{c-x}{b} = 1$; or $\frac{bx}{a} + c - x = b$; or $bx + ac - ax = ab$; or rather, $bx - ax = ab - ac$; whence we have

$$x = \frac{ab - ac}{b - a} \text{ or } x = \frac{a(b-c)}{b-a}.$$

Consequently,

$$c - x = \frac{bc - ab}{b - a} = \frac{b(c-a)}{b-a}.$$

Answer. The banker will give $\frac{a(b-c)}{b-a}$ pieces of the first kind, and $\frac{b(c-a)}{b-a}$ pieces of the second kind.

Remark. These two numbers are easily found by the rule of three, when it is required to apply the results which we have obtained. To find the first we say;

$$b - a : b - c = a : \frac{a(b-c)}{b-a}.$$

The second number is found thus;

$$b - a : c - a = b : \frac{b(c - a)}{b - a}.$$

It ought to be observed also that a is less than b, and that c is also less than b; but at the same time greater than a, as the nature of the thing requires.

521. *Question* XX. A banker has two kinds of change; 10 pieces of one make a crown, and 20 pieces of the other make a crown. Now, a person wishes to change a crown into 17 pieces of money: How many of each must he have?

We have here $a = 10$, $b = 20$, and $c = 17$; which furnishes the following proportions;

I. $10 : 3 = 10 : 3$, so that the number of pieces of the first kind is 3.

II. $10 : 7 = 20 : 14$, and there are 14 pieces of the second kind.

522. *Question* XXI. A father leaves at his death several children, who share his property in the following manner:

The first receives a hundred crowns and the tenth part of the remainder.

The second receives two hundred crowns and the tenth part of what remains.

The third takes three hundred crowns and the tenth part of what remains.

The fourth takes four hundred crowns and the tenth part of what then remains, and so on.

Now it is found at the end that the property has been divided equally among all the children. Required, how much it was, how many children there were, and how much each received?

This question is rather of a singular nature, and therefore deserves particular attention. In order to resolve it more easily, we shall suppose the whole fortune $= z$ crowns; and since all the children receive the same sum, let the share of each $= x$, by which means the number of children is expressed by $\frac{z}{x}$. This being laid down, we may proceed to the solution of the question, which will be as follows:

Sum, or property to be divided.	Order of the Children.	Portion of each.	Differences.
z	1st.	$x = 100 + \frac{z - 100}{10}$	
$z - x$	2d.	$x = 200 + \frac{z - x - 200}{10}$	$100 - \frac{x - 100}{10} = 0$
$z - 2x$	3d.	$x = 300 + \frac{z - 2x - 300}{10}$	$100 - \frac{x - 100}{10} = 0$
$z - 3x$	4th.	$x = 400 + \frac{z - 3x - 400}{10}$	$100 - \frac{x - 100}{10} = 0$
$z - 4x$	5th.	$x = 500 + \frac{z - 4x - 500}{10}$	$100 - \frac{x - 100}{10} = 0$
$z - 5x$	6th.	$x = 600 + \frac{z - 5x - 600}{10}$	and so on.

We have inserted, in the last column, the differences which we obtain by subtracting each portion from that which follows. Now all the portions being equal, each of the differences must be $= 0$. And as it happens that all these differences are expressed exactly alike, it will be sufficient to make one of them equal to nothing, and we shall have the equation $100 - \frac{x - 100}{10} = 0$. Multiplying by 10, we have $1000 - x - 100 = 0$, or $900 - x = 0$; consequently $x = 900$.

We now know, therefore, that the share of each child was 900 crowns; so that taking any one of the equations of the third column, the first, for example, it becomes, by substituting the value of x,

$$900 = 100 + \frac{z - 100}{10},$$

whence we immediately obtain the value of z; for we have

$$9000 = 1000 + z - 100, \text{ or } 9000 = 900 + z;$$

wherefore $z = 8100$; and consequently $\frac{z}{x} = 9$.

Answer. So that the number of children $= 9$; the fortune left by the father $= 8100$ crowns; and the share of each child $= 900$ crowns.

CHAPTER IV.

Of the Resolution of Two or more Equations of the First Degree.

523. It frequently happens that we are obliged to introduce into algebraic calculations two or more unknown quantities, represented by the letters x, y, z; and if the question is determinate, we are brought to the same number of equations; from which, it is then required to deduce the unknown quantities. As we consider, at present, those equations only which contain no powers of an unknown quantity higher than the first, and no products of two, or more unknown quantities, it is evident that these equations will all have the form $az + by + cx = d$.

524. Beginning, therefore, with two equations, we shall endeavour to find from them the values of z and y. That we may consider this case in a general manner, let the two equations be,

$$\text{I. } ax + by = c, \text{ and II. } fx + gy = h,$$

in which a, b, c, and f, g, h, are known numbers. It is required, therefore, to obtain, from these two equations, the two unknown quantities x and y.

525. The most natural method of proceeding will readily present itself to the mind; which is to determine, from both equations, the value of one of the unknown quantities, x for example, and to consider the equality of those two values; for then we shall have an equation in which the unknown quantity y will be found by itself, and may be determined by the rules which we have already given. Knowing y, we have only to substitute its value in one of the quantities that express x.

526. According to this rule, we obtain from the first equation, $x = \frac{c - by}{a}$, and from the second $x = \frac{h - gy}{f}$; stating these two values equal to one another, we have this new equation;

$$\frac{c - by}{a} = \frac{h - gy}{f};$$

multiplying by a, the product is

$$c - by = \frac{ah - agy}{f};$$

multiplying by f, the product is $fc - fby = ah - agy$; adding agy, we have $fc - fby + agy = ah$; subtracting fc there remains

$-fby + agy = ah - fc$; or $(ag - bf)y = ah - fc$; lastly, dividing by $ag - bf$, we have

$$y = \frac{ah - fc}{ag - bf}.$$

In order now to substitute this value of y in one of the two values which we have found of x, as in the first, where $x = \frac{c - by}{a}$, we shall first have

$$-by = -\frac{abh + bcf}{ag - bf};$$

whence

$$c - by = c - \frac{abh + bcf}{ag - bf},$$

or

$$c - by = \frac{acg - bcf - abh + bcf}{ag - bf} = \frac{acg - abh}{ag - bf};$$

and dividing by a,

$$x = \frac{c - by}{a} = \frac{cg - bh}{ag - bf}.$$

527. *Question* I. To illustrate this method by examples let it be proposed to find two numbers, whose sum may be $= 15$, and difference $= 7$.

Let us call the greater number x, and the less y. We shall have,

$$\text{I. } x + y = 15, \text{ and II. } x - y = 7.$$

The first equation gives $x = 15 - y$, and the second $x = 7 + y$; whence results the new equation $15 - y = 7 + y$. So that $15 = 7 + 2y$; $2y = 8$, and $y = 4$; by which means we find $x = 11$.

Answer. The less number is 4, and the greater is 11.

528. *Question* II. We may also generalize the preceding question, by requiring two numbers, whose sum may be $= a$, and the difference $= b$.

Let the greater of the two be $= x$, and the less $= y$.

We shall have I. $x + y = a$, and II. $x - y = b$; the first equation gives $x = a - y$; and the second $x = b + y$.

Wherefore $a - y = b + y$; $a = b + 2y$; $2y = a - b$; lastly, $y = \frac{a-b}{2}$, and consequently,

$$x = a - y = a - \frac{a-b}{2} = \frac{a+b}{2}.$$

Answer. The greater number, or x, is $= \frac{a+b}{2}$, and the less, or y, is $= \frac{a-b}{2}$, or which comes to the same, $x = \frac{1}{2}a + \frac{1}{2}b$, and $y = \frac{1}{2}a - \frac{1}{2}b$; and hence we derive the following theorem. *When the sum of any two numbers is* a, *and their difference is* b, *the greater of the two numbers will be equal to half the sum* plus *half the difference: and the less of the two numbers will be equal to half the sum* minus *half the difference.*

529. We may also resolve the same question in the following manner:

Since the two equations are $x + y = a$, and $x - y = b$; if we add one to the other, we have $2x = a + b$. Wherefore

$$x = \frac{a+b}{2}.$$

Lastly, subtracting the same equation from the other, we have $2y = a - b$; wherefore $y = \frac{a-b}{2}$.

530. *Question* III. A mule and an ass were carrying burdens amounting to some hundred weight. The ass complained of his, and said to the mule, I need only one hundred weight of your load, to make mine twice as heavy as yours. The mule answered, Yes, but if you gave me a hundred weight of yours, I should be loaded three times as much as you would be. How many hundred weight did each carry?

Suppose the mule's load to be x hundred weight, and that of the ass to be y hundred weight. If the mule gives one hundred weight to the ass, the one will have $y + 1$, and there will remain for the other $x - 1$; and since, in this case, the ass is loaded twice as much as the mule, we have $y + 1 = 2x - 2$.

Further, if the ass gives a hundred weight to the mule, the latter has $x + 1$, and the ass retains $y - 1$; but the burden of the former being now three times that of the latter, we have,

$$x + 1 - 3y - 3.$$

Our two equations will consequently be,

I. $y + 1 = 2x - 2$, II. $x + 1 = 3y - 3$.

The first gives $x = \frac{y+3}{2}$, and the second gives $x = 3y - 4$; whence we have the new equation $\frac{y+3}{2} = 3y - 4$, which gives $y = \frac{11}{5}$, and also determines the value of x, which becomes $2\frac{3}{5}$.

Answer. The mule carried $2\frac{3}{5}$ hundred weight, and the ass carried $2\frac{1}{5}$ hundred weight.

531. When there are three unknown numbers, and as many equations; as, for example, I. $x + y - z = 8$, II. $x + z - y = 9$, III. $y + z - x = 10$, we begin, as before, by deducing a value of x from each, and we have from the Ist, $x = 8 + z - y$; from the II^{d}, $x = 9 + y - z$; and from the III^{d}, $x = y + z - 100$.

Comparing the first of these values with the second, and after that with the third also, we have the following equations:

I. $8 + z - y = 9 + y - z$, II. $8 + z - y = y + z - 10$.

Now, the first gives $2z - 2y = 1$, and the second gives $2y = 18$, or $y = 9$; if therefore we substitute this value of y in $2z - 2y = 1$, we have $2z - 18 = 1$, and $2z = 19$, so that $z = 9\frac{1}{2}$; it remains therefore only to determine x, which is easily found $= 8\frac{1}{2}$.

Here it happens that the letter z vanishes in the last equation, and that the value of y is found immediately. If this had not been the case, we should have had two equations between z and y, to be resolved by the preceding rule.

532. Suppose we had found the three following equations.

$$\text{I. } 3x + 5y - 4z = 25,\quad \text{II. } 5x - 2y + 3z = 46,$$
$$\text{III. } 3y + 5z - x = 62.$$

If we deduce from each the value of x, we shall have

$$\text{I. } x = \frac{25 - 5y + 4z}{3},\quad \text{II. } x = \frac{46 + 2y - 3z}{5},$$
$$\text{III. } x = 3y + 5z - 62.$$

Comparing these three values together, and first the third with the first, we have

$$3y + 5z - 62 = \frac{25 - 5y + 4z}{3};$$

multiplying by 3, $9y + 15z - 186 = 25 - 5y + 4z$; so that $9y + 15z = 211 - 5y + 4z$, and $14y + 11z = 211$ by the first and the third. Comparing also the third with the second, we have

$$3y + 5z - 62 = \frac{46 + 2y - 3z}{5},$$

or $46 + 2y - 3z = 15y + 25z - 310$, which when reduced is $356, = 13y + 28z$.

We shall now deduce, from these two new equations, the value of y;

I. $211 = 14y + 11z$; wherefore $14y = 211 - 11z$, and

$$y = \frac{211 - 11z}{14}.$$

II. $356 = 13y + 28z$; wherefore $13y = 356 - 28z$, and

$$y = \frac{356 - 28z}{13}.$$

These two values form the new equation

$$\frac{211 - 11z}{14} = \frac{356 - 28z}{13},$$

which becomes $2743 - 143z = 4984 - 392z$, or $249z = 2241$, whence $z = 9$.

This value being substituted in one of the two equations of y and z, we find $y = 8$; and lastly a similar substitution, in one of the three values of x, will give $x = 7$.

533. If there were more than three unknown quantities to be determined, and as many equations to be resolved, we should proceed in the same manner; but the calculations would often prove very tedious.

It is proper, therefore, to remark, that, in each particular case, means may always be discovered of greatly facilitating its resolution. These means consist in introducing into the calculation, beside the principal unknown quantities, a new unknown quantity arbitrarily assumed, such as, for example, the sum of all the rest; and when a person is a little practised in such calculations he easily perceives what is most proper to do. The following examples may serve to facilitate the application of these artifices.

534. *Question* IV. Three persons play together; in the first game, the first loses to each of the other two, as much money as

each of them has. In the next, the second person loses to each of the other two, as much money as they have already. Lastly, in the third game, the first and the second person gain each, from the third, as much money as they had before. They then leave off, and find that they have all an equal sum, namely 24 louis each. Required, with how much money each sat down to play?

Suppose that the stake of the first person was x louis, that of the second y, and that of the third z. Further, let us make the sum of all the stakes, or $x + y + z = s$. Now, the first person losing in the first game as much money as the other two have, he loses $s - x$; (for he himself having had x, the two others must have had $s - x$); wherefore there will remain to him $2x - s$; the second will have $2y$, and the third will have $2z$.

So that, after the first game, each will have as follows:

the I. $2x - s$, the II. $2y$, the III. $2z$.

In the second game, the second person, who has now $2y$, loses as much money as the other two have, that is to say $s - 2y$; so that he has left $4y - s$. With regard to the others, they will each have double what they had; so that after the second game, the three persons have;

the I. $4x - 2s$, the II. $4y - s$, the III. $4z$.

In the third game, the third person, who has now $4z$, is the loser; he loses to the first $4x - 2s$, and to the second $4y - s$; consequently after this game the three persons will have;

the I. $8x - 4s$, the II. $8y - 2s$, the III. $8z - s$.

Now, each having at the end of this game 24 louis, we have three equations, the first of which immediately gives x, the second y, and the third z; further, s is known to be $= 72$, since the three persons have in all 72 louis at the end of the last game; but it is not necessary to attend to this at first. We have

I. $8x - 4s = 24$, or $8x = 24 + 4s$, or $x = 3 + \frac{1}{2}s$;
II. $8y - 2s = 24$, or $8y = 24 + 2s$, or $y = 3 + \frac{1}{4}s$;
III. $8z - s = 24$, or $8z = 24 + s$, or $z = 3 + \frac{1}{8}s$;

Adding these three values, we have

$$x + y + z = 9 + \frac{7}{8}s.$$

So that, since $x + y + z = s$, we have $s = 9 + \frac{7}{8}s$; wherefore $\frac{1}{8}s = 9$, and $s = 72$.

If we now substitute this value of s in the expressions which we have found for x, y, and z, we shall find that before they began to play, the first person had 39 louis; the second 21 louis; and the third 12 louis.

This solution shows, that by means of an expression for the sum of the three unknown quantities, we may overcome the difficulties which occur in the ordinary method.

535. Although the preceding question appears difficult at first, it may be resolved even without algebra. We have only to try to do it inversely. Since the players, when they left off, had each 24 louis, and, in the third game, the first and the second doubled the money, they must have had before that last game;

The I. 12, the II. 12, and the III. 48.

In the second game the first and the third doubled their money; so that before that game they had;

The I. 6, the II. 42, and the III. 24.

Lastly, in the first game, the second and the third gained each as much money as they began with; so that at first the three persons had;

I. 39, II. 21, III. 12.

The same result as we obtained by the former solution.

536. *Question* V. Two persons owe 29 pistoles; they have both money, but neither of them enough to enable him, singly to discharge this common debt; the first debtor says therefore to the second, if you give me $\frac{2}{3}$ of your money, I singly will immediately pay the debt. The second answers, that he also could discharge the debt, if the other would give him $\frac{3}{4}$ of his money. Required, how many pistoles each had?

Suppose that the first has x pistoles, and that the second has y pistoles.

We shall first have, $x + \frac{2}{3}y = 29$;
then also, $y + \frac{3}{4}x = 29$.

The first equation gives $x = 29 - \frac{2}{3}y$, and the second,

$$x = \frac{116 - 4y}{3}; \text{ so that } 29 - \frac{2}{3}y = \frac{116 - 4y}{3}.$$

From this equation, we get $y = 14\frac{1}{2}$; wherefore $x = 19\frac{1}{3}$.

Answer. The first debtor had $19\frac{1}{3}$ pistoles, and the second had $14\frac{1}{2}$ pistoles.

537. *Question* VI. Three brothers bought a vineyard for a hundred louis. The youngest says, that he could pay for it alone, if the second gave him half the money which he had; the second says, that if the eldest would give him only the third of his money, he could pay for the vineyard singly; lastly, the eldest asks only a fourth part of the money of the youngest, to pay for the vineyard himself. How much money had each?

Suppose the first had x louis; the second, y louis; the third, z louis; we shall then have the three following equations;

I. $x + \frac{1}{2}y = 100$. II. $y + \frac{1}{3}z = 100$. III. $z + \frac{1}{4}x = 100$; two of which only give the value of x, namely,

$$\text{I. } x = 100 - \tfrac{1}{2}y. \quad \text{III. } x = 400 - 4z.$$

So that we have the equation,

$$100 - \tfrac{1}{2}y = 400 - 4z, \text{ or } 4z - \tfrac{1}{2}y = 300,$$

which must be combined with the second, in order to determine y and z. Now the second equation was $y + \frac{1}{3}z = 100$; we therefore deduce from it $y = 100 - \frac{1}{3}z$; and the equation found last being $4z - \frac{1}{2}y = 300$, we have $y = 8z - 600$. Consequently the final equation is, $100 - \frac{1}{3}z = 8z - 600$; so that $8\frac{1}{3}z = 700$, or $\frac{25}{3}z = 700$, and $z = 84$. Wherefore

$$y = 100 - 28 = 72, \text{ and } x = 64.$$

Answer. The youngest had 64 louis, the second had 72 louis, and the eldest had 84 louis.

538. As, in this example, each equation contains only two unknown quantities, we may obtain the solution required in an easier way.

The first equation gives $y = 200 - 2x$; so that y is determined by x; and if we substitute this value in the second equation, we have $200 - 2x + \frac{1}{3}z = 100$; wherefore $\frac{1}{3}z = 2x - 100$, and $z = 6x - 300$.

So that z is also determined by x; and if we introduce this value into the third equation, we obtain $6x - 300 + \frac{1}{4}x = 100$, in which x stands alone, and which, when reduced to $25x - 1600 = 0$, gives $x = 64$. Consequently, $y = 200 - 128 = 72$, and

$$z = 384 - 300 = 84.$$

539. We may follow the same method, when we have a greater number of equations. Suppose, for example, that we have in general:

$$\text{I. } u + \frac{x}{a} = n, \text{ II. } x + \frac{y}{b} = n, \text{ III. } y + \frac{z}{c} = n,$$

$$\text{IV. } z + \frac{u}{d} = n; \text{ or, reducing the fractions,}$$

$$\text{I. } au + x = an, \text{ II. } bx + y = bn, \text{ III. } cy + z = cn,$$
$$\text{IV. } dz + u = dn.$$

Here, the first equation gives immediately $x = an - au$, and this value being substituted in the second, we have

$abn - abu + y = bn$; so that $y = bn - abn + abu$;

the substitution of this value in the third equation, gives

$$bcn - abcn + abcu + z = cn;$$

wherefore $z = cn - bcn + abcn - abcu$; substituting this in the fourth equation, we have

$$cdn - bcdn + abcdn - abcdu + u = dn.$$

So that $dn - cdn + bcdn - abcdn = -abcdu + u$, or $(abcd - 1)u = abcdn - bcdn + cdn - dn$; whence we have

$$u = \frac{abcdn - bcdn + cdn - dn}{abcd - 1} = n \times \frac{(abcd - bcd + cd - d).}{abcd - 1}$$

Consequently, we shall have,

$$x = \frac{abcdn - acdn + adn - an}{abcd - 1} = n \times \frac{(abcd - acd + ad - a).}{abcd - 1}$$

$$y = \frac{abcdn - abdn + abn - bn}{abcd - 1} = n \times \frac{(abcd - abd + ab - b).}{abcd - 1}$$

$$z = \frac{abcdn - abcn + bcn - cn}{abcd - 1} = n \times \frac{(abcd - abc + bc - c).}{abcd - 1}$$

$$u = \frac{abcdn - bcdn + cdn - dn}{abcd - 1} = n \times \frac{(abcd - bcd + cd - d).}{abcd - 1}$$

540. *Question* VII. A captain has three companies, one of Swiss, another of Swabians, and a third of Saxons. He wishes to storm with part of these troops, and he promises a reward of 901 crowns, on the following condition:

That each soldier of the company which assaults, shall receive 1 crown, and that the rest of the money shall be equally distributed among the two other companies.

Now it is found, that if the Swiss make the assault, each soldier of the other companies receives $\frac{1}{2}$ of a crown; that, if the Swabians assault, each of the others receives $\frac{1}{3}$ of a crown; lastly, that if the Saxons make the assault, each of the others receives $\frac{1}{4}$ of a crown. Required the number of men in each company?

Let us suppose the number of Swiss $= x$, that of Swabians $= y$, and that of the Saxons $= z$. And let us also make $x + y + z = s$, because it is easy to see, that by this, we abridge the calculation considerably. If, therefore, the Swiss make the assault, their number being x, that of the other will be $s - x$; now, the former receive 1 crown, and the latter half a crown; so that we shall have,

$$x + \tfrac{1}{2}s - \tfrac{1}{2}x = 901.$$

We find in the same manner, that if the Swabians make the assault, we have,

$$y + \tfrac{1}{3}s - \tfrac{1}{3}y = 901.$$

And lastly, that if the Saxons mount the assault, we have

$$z + \tfrac{1}{4}s - \tfrac{1}{4}z = 901.$$

Each of these three equations will enable us to determine one of the unknown quantities x, y, z;

For the first gives $x = 1802 - s$,
the second gives $2y = 2703 - s$,
the third gives $3z = 3604 - s$,

If we now take the values of $6x$, $6y$, and $6z$, and write those values one above the other, we shall have,

$$6x = 10812 - 6s,$$
$$6y = 8109 - 3s,$$
$$6z = 7208 - 2s,$$

and adding; $6s = 26129 - 11s$, or $17s = 26129$; so that $s = 1537$; this is the whole number of soldiers, by which means we find,

$$x = 1802 - 1537 = 265;$$
$$2y = 2703 - 1537 = 1166, \text{ or } y = 583;$$
$$3z = 3604 - 1537 = 2067, \text{ or } z = 689.$$

Answer. The company of Swiss consists of 265 men; that of Swabians 583; and that of Saxons 689.

CHAPTER V.

Of the Resolution of Pure Quadratic Equations.

541. An *equation is said to be of the second degree, when it contains the square or the second power of the unknown quantity, without any of its higher powers.* An equation, containing likewise the third power of the unknown quantity, belongs to cubic equations, and its resolution requires particular rules. There are, therefore, only three kinds of terms in an equation of the second degree.

1. The terms in which the unknown quantity is not found at all, or which are composed only of known numbers.

2. The terms in which we find only the first power of the unknown quantity.

3. The terms which contain the square, or the second power of the unknown quantity.

So that x signifying an unknown quantity, and the letters a, b, c, d, &c. representing known numbers, the terms of the first kind will have the form a, the terms of the second kind will have the form $b\,x$, and the terms of the third kind will have the form $c\,x\,x$.

542. We have already seen, how two or more terms of the same kind may be united together, and considered as a single term.

For example, we may consider the formula $a\,x\,x - b\,x\,x + c\,x\,x$ as a single term, representing it thus $(a - b + c)\,x\,x$; since, in fact, $(a - b + c)$ is a known quantity.

And also, when such terms are found on both sides of the sign $=$, we have seen how they may be brought to one side, and then reduced to a single term. Let us take, for example, the equation,

$$2\,x\,x - 3\,x + 4 = 5\,x\,x - 8\,x + 11;$$

We first subtract $2\,x\,x$, and there remains

$$-3\,x + 4 = 3\,x\,x - 8\,x + 11;$$

then adding $8\,x$, we obtain,

$$5x + 4 = 3xx + 11;$$

Lastly, subtracting 11, there remains $3xx = 5x - 7$.

543. We may also bring all the terms to one side of the sign $=$, so as to leave only 0 on the other. It must be remembered, however, that when terms are transposed from one side to the other, their signs must be changed.*

Thus, the above equation will assume this form,

$$3xx - 5x + 7 = 0;$$

and, for this reason also, *the following general formula represents all equations of the second degree.*

$$axx \pm bx \pm c = 0,$$

in which the sign $\pm$ is read *plus* or *minus*, and indicates that such terms may be sometimes positive and sometimes negative.

544. Whatever be the original form of a quadratic equation, it may always be reduced to this formula of three terms. If we have, for example, the equation

$$\frac{ax + b}{cx + d} = \frac{cx + f}{gx + h},$$

we must, first, reduce the fractions; multiplying, for this purpose, by $cx + d$, we have

$$ax + b = \frac{cexx + cfx + edx + fd}{gx + h},$$

then by $gx + h$, we have

$$agxx + bgx + ahx + bh = cexx + cfx + edx + fd,$$

which is an equation of the second degree, and reducible to the three following terms, which we shall transpose by arranging them in the usual manner:

$$\begin{aligned} 0 = agxx &+ bgx + bh, \\ - cexx &+ ahx - fd, \\ &- cfx, \\ &- edx. \end{aligned}$$

We may exhibit this equation also in the following form, which is still more clear:

$$0 = (ag - ce)xx + (bg + ah - cf - ed)x + bh - fd.$$

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* That is, the quantity thus transposed is added to or subtracted from each side of the equation.

545. Fquations of the second degree, in which all the three kinds of terms are found, are called *complete*; and the resolution of them is attended with greater difficulties; for which reason we shall first consider those, in which one of the terms is wanting.

Now, if the term xx were not found in the equation, it would not be a quadratic, but would belong to those of which we have already treated. *If the term, which contains only known numbers, were wanting, the equation would have this form,* $axx \pm bx = 0$, *which being divisible by* x, *may be reduced to* $ax \pm b = 0$, *which is likewise a simple equation, and belongs not to the present class.*

546. *But when the middle term, which contains the first power of* x, *is wanting, the equation assumes this form,* $axx \pm c = 0$, *or* $axx = \mp c$; as the sign of *c* may be either positive or negative.

We shall call such an equation a *pure* equation of the second degree, since the resolution of it is attended with no difficulty; for *we have only to divide by* a, *which gives* $xx = \frac{c}{a}$; *and taking the square root of both sides, we find* $x = \sqrt{\frac{c}{a}}$; *by means of which the equation is resolved.*

547. But there are three cases to be considered here. In the first, *when* $\frac{c}{a}$ *is a square number* (of which we can therefore really assign the root) *we obtain for the value of* x *a rational number, which may be either integer or fractional.* For example, the equation $xx = 144$ gives $x = 12$. And $xx = \frac{9}{16}$ gives $x = \frac{3}{4}$.

The second variety is *when* $\frac{c}{a}$ *is not a square,* in which case *we must* therefore *be contented with the sign* $\surd$. If, for example, $xx = 12$, we have $x = \sqrt{12}$, the value of which may be determined by approximation, as we have already shown.

The third case is that in which $\frac{c}{a}$ *becomes a negative number; then the value of* x *is altogether impossible and imaginary; and this result proves that the question, which leads to such an equation, is in itself impossible.*

548. We shall also observe before proceeding further, that whenever it is required to extract the square root of a number, that root, as we have already remarked, has always two values, the one positive and the other negative. *Suppose we have the equation* $xx = 49$,

the value of x *will be not only* $+ 7$, *but also* $- 7$, *which is expressed by* $x = \pm 7$. So that all those questions admit of a double answer; but it will be easily perceived that in several cases, in those, for example, which relate to a certain number of men, the negative value cannot exist.

549. In such equations also, as $a x x = b x$, where the known quantity c is wanting, there may be two values of x, though we find only one if we divide by x. In the equation $x x = 3 x$, for example, in which it is required to assign such a value of x, that $x x$ may become equal to $3 x$, this is done by supposing $x = 3$, a value which is found by dividing the equation by x; but beside this value, there is also another, which is equally satisfactory, namely, $x = 0$; for then $x x = 0$, and $3 x = 0$. *Equations*, therefore, *of the second degree, in general, admit of two solutions, whilst simple equations admit only of one.*

We shall now illustrate, by some examples, what we have said with regard to pure equations of the second degree.

550. *Question* I. Required a number, the half of which multiplied by the third may produce 24.

Let this number $= x$; $\frac{1}{2} x$, multiplied by $\frac{1}{3} x$, must give 24; we shall therefore have the equation $\frac{1}{6} x x = 24$.

Multiplying by 6, we have $x x = 144$; and the extraction of the root gives $x = \pm 12$. We put $\pm$; for if $x = + 12$, we have $\frac{1}{2} x = 6$, and $\frac{1}{3} x = 4$; now the product of these two numbers is 24; and if $x = - 12$, we have $\frac{1}{2} x = - 6$, and $\frac{1}{3} = - 4$, the product of which is likewise 24.

551. *Question* II. Required a number such, that by adding 5 to it, and subtracting 5 from it, the product of the sum by the difference would be 96.

Let this number be x, then $x + 5$, multiplied by $x - 5$, must give 96; whence results the equation, $x x - 25 = 96$.

Adding 25, we have $x x = 121$; and extracting the root, we have $x = 11$. Thus $x + 5 = 16$, also $x - 5 = 6$; and lastly, $6 \times 16 = 96$.

552. *Question* III. Required a number such, that by adding it to 10, and subtracting it from 10, the sum, multiplied by the remainder, or difference, will give 51.

Let x be this number; $10 + x$, multiplied by $10 - x$, must make 51, so that $100 - x x = 51$. Adding $x x$, and subtracting 51, we have $x x = 49$, the square root of which gives $x = 7$.

553. *Question* IV. Three persons, who had been playing, leave off; the first, with as many times 7 crowns, as the second has 3 crowns; and the second, with as many times 17 crowns, as the third has 5 crowns. Further, if we multiply the money of the first by the money of the second, and the money of the second by the money of the third, and lastly, the money of the third by that of the first, the sum of these three products will be $3830\frac{2}{3}$. How much money has each?

Suppose that the first player has x crowns; and since he has as many times 7 crowns, as the second has 3 crowns, we know that his money is to that of the second, in the ratio of 7 : 3.

We shall therefore make $7 : 3 = x$ to the money of the second player, which is therefore $\frac{3}{7} x$.

Further, as the money of the second player is to that of the third in the ratio of 17 : 5, we shall say, $17 : 5 = \frac{3}{7} x$ to the money of the third player, or to $\frac{15}{119} x$.

Multiplying x, or the money of the first player, by $\frac{3}{7} x$, the money of the second, we have the product $\frac{3}{7} x x$. Then $\frac{3}{7} x$, the money of the second, multiplied by the money of the third, or by $\frac{15}{119} x$, gives $\frac{45}{833} x x$. Lastly, the money of the third, or $\frac{15}{119} x$ multiplied by x, or the money of the first, gives $\frac{15}{119} x x$. The sum of these three products is $\frac{3}{7} x x + \frac{45}{833} x x + \frac{15}{119} x x$; and, reducing these fractions to the same denominator, we find their sum $\frac{507}{833} x x$, which must be equal to the number $3830\frac{2}{3}$.

We have, therefore, $\frac{507}{833} x x = 3830\frac{2}{3}$.

So that $\frac{1521}{833} x x = 11492$, and $1521 x x$ being equal to 9572836, dividing by 1521, we have $x x = \frac{9572836}{1521}$; and taking its root, we find $x = \frac{3094}{39}$. This fraction is reducible to lower terms if we divide by 13; so that $x = \frac{238}{3} = 79\frac{1}{3}$; and hence we conclude, that $\frac{3}{7} x = 34$, and $\frac{15}{119} x = 10$.

Answer. The first player has $79\frac{1}{3}$ crowns, the second has 34 crowns, and the third 10 crowns.

Remark. This calculation may be performed in an easier manner; namely, by taking the factors of the numbers which present themselves, and attending chiefly to the squares of those factors.

It is evident, that $507 = 3 \times 169$, and that 169 is the square of 13; then, that $833 = 7 \times 119$, and $119 = 7 \times 17$. Now we have $\frac{3 \times 169}{17 \times 49} x x = 3830\frac{2}{3}$, and if we multiply by 3, we have

$\frac{9 \times 169}{17 \times 49} xx = 11492$. Let us resolve this number also into its factors; we readily perceive, that the first is 4, that is to say, that $11492 = 4 \times 2873$; further, 2873 is divisible by 17; so that $2873 = 17 \times 169$. Consequently our equation will assume the following form; $\frac{9 \times 169}{17 \times 49} xx = 4 \times 17 \times 169$, which, divided by 169, is reduced to $\frac{9}{17 \times 49} xx = 4 \times 17$; multiplying also by 17×49, and dividing by 9, we have $xx = \frac{4 \times 289 \times 49}{9}$, in which all the factors are squares; whence we have, without any further calculation, the root

$$x = \frac{2 \times 17 \times 7}{3} = \frac{238}{3} = 79\tfrac{1}{3},$$

as before.

554. *Question* V. A company of merchants appoint a factor at Archangel. Each of them contributes for the trade, which they have in view, ten times as many crowns as there are partners. The profit of the factor is fixed at twice as many crowns *per cent.*, as there are partners. Further, if we multiply the $\frac{1}{100}$ part of his total gain by $2\frac{2}{9}$, the number of partners will be found. Required, what that number is.

Let it be $= x$; and since each partner has contributed $10x$, the whole capital is $= 10xx$. Now, for every hundred crowns, the factor gains $2x$, so that with the capital of $10xx$ his profit will be $\frac{1}{5}x^3$. The $\frac{1}{100}$ part of this gain is $\frac{1}{500}x^3$; multiplying by $2\frac{2}{9}$, or by $\frac{20}{9}$, we have $\frac{20}{4500}x^3$, or $\frac{1}{225}x^3$, and this must be equal to the number of partners, or x.

We have, therefore, the equation $\frac{1}{225}x^3 = x$, or $x^3 = 225x$; which appears, at first, to be of the third degree; but as we may divide by x, it is reduced to the quadratic $xx = 225$, whence $x = 15$.

Answer. There are fifteen partners, and each contributed 150 crowns.

CHAPTER VI.

Of the Resolution of Mixt Equations of the Second Degree.

555. An *equation of the second degree is said to be* mixt, *or* complete,* *when three kinds of terms are found in it, namely, that which contains the square of the unknown quantity, as* a x x; *that, in which the unknown quantity is found only of the first power, as* b x; *lastly, the kind of terms which is composed only of known quantities.* And since we may unite two or more terms of the same kind into one, and bring all the terms to one side of the sign =, the general form of a mixt equation of the second degree will be

$$a x x \mp b x \mp c = 0.$$

In this chapter, we shall show how the value of x is derived from such equations. It will be seen that there are two methods of obtaining it.

556. An equation of the kind that we are now considering may be reduced, by division, to such a form, that the first term may contain only the square $x x$ of the unknown quantity x. We shall leave the second term on the same side with x, and transpose the known term to the other side of the sign =. By these means our equation will assume the form $x x \pm p x = \pm q$, in which p and q represent any known numbers, positive or negative; and the whole is at present reduced to determining the true value of x. We shall begin with remarking, that if $x x + p x$ were a real square, the resolution would be attended with no difficulty, because it would only be required to take the square root of both sides.

557. But it is evident that $x x + p x$ cannot be a square; since we have already seen, that *if a root consists of two terms, for example,* x + n, *its square always contains three terms, namely, twice the product of the two parts, besides the square of each part; that is to say, the square of* x + n *is* x x + 2 n x + n n. Now we have already on one side $x x + p x$; *we may,* therefore, *consider* x x *as the square of the first part of the root, and in this case* p x *must represent twice the product of* x, *the first part of the root by the*

* Sometimes called also *affected.*

second part; consequently, this second part must be $\frac{1}{2}$ p, *and in fact the square of* $x + \frac{1}{2}p$, *is found to be* $xx + px + \frac{1}{4}pp$.

558. *Now* $xx + px + \frac{1}{4}pp$ *being a real square, which has for its root,* $x + \frac{1}{2}p$, *if we resume our equation* $xx + px = q$, *we have only to add* $\frac{1}{4}pp$ *to both sides, which gives us*

$$xx + px + \frac{1}{4}pp = q + \frac{1}{4}pp,$$

the first side being actually a square, and the other containing only known quantities. If, therefore, we take the square root of both sides, we find

$$x + \frac{1}{2}p = \sqrt{(\tfrac{1}{4}pp + q)};$$

and subtracting $\frac{1}{2}$ p, *we obtain*

$$x = -\frac{1}{2}p + \sqrt{(\tfrac{1}{4}pp + q)};$$

and, as every square root may be taken either affirmatively or negatively, we shall have for x *two values expressed thus;*

$$x = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + q}.$$

559. This formula contains the rule by which all quadratic equations may be resolved, and it will be proper to commit it to memory, that it may not be necessary to repeat, every time, the whole operation which we have gone through. We may always arrange the equation, in such a manner, that the pure square xx may be found on one side, and the above equation have the form $xx + px = q$, where we see immediately that

$$x = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + q}.$$

560. The general rule, therefore, which we deduce from this, in order to resolve the equation $xx = -px + q$, is founded on this consideration:

That the unknown quantity x is equal to half the coefficient, or multiplier of x on the other side of the equation, *plus* or *minus* the square root of the square of this number, and the known quantity which forms the third term of the equation.

Thus if we had the equation $xx = 6x + 7$, we should immediately say, that $x = 3 \pm \sqrt{9 + 7} = 3 \pm 4$, whence we have these two values of x, I. $x = 7$; II. $x = -1$. In the same manner, the equation $xx = 10x - 9$, would give

$$x = 5 \pm \sqrt{25-9} = 5 \pm 4,$$

that is to say, the two values of x are 9 and 1.

561. This rule will be still better understood, by distinguishing the following cases. I. when p is an even number; II. when p is an odd number; and III. when p is a fractional number.

I. Let p be an even number, and the equation such, that

$$x x = 2 p x + q;$$

we shall, in this case, have $x = p \pm \sqrt{p p + q}$.

II. Let p be an odd number, and the equation $x x = p x + q$; we shall here have

$$x = \tfrac{1}{2} p \pm \sqrt{\frac{1}{4} p p + q};$$

and since

$$\tfrac{1}{4} p p + q = \frac{p p + 4 q}{4},$$

we may extract the square root of the denominator, and write

$$x = \tfrac{1}{2} p \pm \frac{\sqrt{p p + 4 q}}{2} = \frac{p \pm \sqrt{p p + 4 q}}{2}.$$

III. Lastly, if p be a fraction, the equation may be resolved in the following manner; let the equation be

$$a x x = b x + c, \text{ or } x x = \frac{b x}{a} + \frac{c}{a},$$

and we shall have by the rule,

$$x = \frac{b}{2 a} \pm \sqrt{\frac{b b}{4 a a} + \frac{c}{a}}.$$

Now,

$$\frac{b b}{4 a a} + \frac{c}{a} = \frac{b b + 4 a c}{4 a a},$$

the denominator of which is a square; so that

$$x = \frac{b \pm \sqrt{b b + 4 a c}}{2 a}.$$

562. The other method of resolving mixt quadratic equations, is to transform them into pure equations. This is done by substitution; for example, in the equation $x x = p x + q$, instead of the unknown quantity x, we may write another unknown quantity y, such

that $x = y + \frac{1}{2}p$; by which means, when we have determined y, we may immediately find the value of x.

If we make this substitution of $y + \frac{1}{2}p$ instead of x, we have $xx = yy + py + \frac{1}{4}pp$, and $px = py + \frac{1}{2}pp$; consequently our equation will become $yy + py + \frac{1}{4}pp = py + \frac{1}{2}pp + q$, which is first reduced, by subtracting py, to

$$yy + \tfrac{1}{4}pp = \tfrac{1}{2}pp + q;$$

and then, by subtracting $\frac{1}{4}pp$, to $yy = \frac{1}{4}pp + q$. This is a pure quadratic equation, which immediately gives

$$y = \pm\sqrt{\frac{1}{4}pp + q}.$$

Now, since $x = y + \frac{1}{2}p$, we have

$$x = \tfrac{1}{2}p \pm \sqrt{\frac{1}{4}pp + q},$$

as we found it before. We have only, therefore, to illustrate this rule by some examples.

563. *Question* I. There are two numbers; one exceeds the other by 6, and their product is 91. What are those numbers?

If the less is x, the other is $x + 6$, and their product

$$xx + 6x = 91.$$

Subtracting $6x$, there remains $xx = 91 - 6x$, and the rule gives $x = -3 \pm \sqrt{9 + 91} = -3 \pm 10$; so that $x = 7$, and $x = -13$.

Answer. The question admits of two solutions;

By one, the less number x is $= 7$, and the greater $x + 6 = 13$.

By the other, the less number $x = -13$, and the greater $x + 6 = -7$.

564. *Question* II. To find a number such, that if 9 be taken from its square, the remainder may be a number, as many units greater than 100, as the number sought is less than 23.

Let the number sought $= x$; we know, that $xx - 9$ exceeds 100 by $xx - 109$. And since x is less than 23 by $23 - x$, we have this equation; $xx - 109 = 23 - x$.

Wherefore $xx = -x + 132$, and, by the rule,

$$x = -\tfrac{1}{2} \pm \sqrt{\frac{1}{4} + 132} = -\frac{1}{2} \pm \sqrt{\frac{529}{4}} = -\frac{1}{2} \pm \frac{23}{2}.$$

So that $x = 11$, and $x = -12$.

Answer. When only a positive number is required, that number will be 11, the square of which *minus* 9 is 112, and consequently greater than 100 by 12, in the same manner as 11 is less than 23 by 12.

565. *Question* III. To find a number such, that if we multiply its half by its third, and to the product add half the number required, the result will be 30.

Suppose that number $= x$, its half, multiplied by its third, will make $\frac{1}{6}\,x\,x$; so that $\frac{1}{6}\,x\,x + \frac{1}{2}\,x = 30$. Multiplying by 6, we have $x\,x + 3\,x = 180$, or $x\,x = -3\,x + 180$, which gives

$$x = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + 180} = -\frac{3}{2} \pm \frac{27}{2}.$$

Consequently x is either $= 12$, or -15.

566. *Question* IV. To find two numbers in a double ratio to each other, and such that if we add their sum to their product, we obtain 90.

Let one of the numbers $= x$, then the other will be $= 2\,x$; their product also $= 2\,x\,x$, and if we add to this $3\,x$, or their sum, the new sum ought to make 90. So that

$2\,x\,x + 3\,x = 90$; $2\,x\,x = 90 - 3\,x$; $x\,x = -\frac{3}{2}\,x + 45$, whence we obtain

$$x = -\frac{3}{4} \pm \sqrt{\frac{9}{16} + 45} = -\frac{3}{4} \pm \frac{27}{4}.$$

Consequently $x = 6$, or $-7\frac{1}{2}$.

567. *Question* V. A horse-dealer, who bought a horse for a certain number of crowns, sells it again for 119 crowns, and his profit is as much per cent. as the horse cost him. Required, what he gave for it?

Suppose the horse cost x crowns; then as the horse-dealer gains x per cent., we shall say, if 100 give the profit x, what does x give?

Answer, $\frac{x\,x}{100}$. Since, therefore, he has gained $\frac{x\,x}{100}$, and the horse originally cost him x crowns, he must have sold it for $x + \frac{x\,x}{100}$; wherefore

$$x - \frac{x\,x}{100} = 119.$$

Subtracting x, we have

$$\frac{xx}{100} = -x + 119;$$

and multiplying by 100, we have $xx = -100x + 11900$. Applying the rule we find

$x = -50 \pm \sqrt{2500 + 11900} = -50 \pm \sqrt{14400} = -50 \pm 120.$

Answer. The horse cost 70 crowns, and since the horse-dealer gained 70 per cent. when he sold it again, the profit must have been 49 crowns. The horse must have been, therefore, sold again for 70 + 49, that is to say, for 119 crowns.

568. *Question* VI. A person buys a certain number of pieces of cloth; he pays, for the first, 2 crowns; for the second, 4 crowns; for the third, 6 crowns, and in the same manner always 2 crowns more for each following piece. Now, all the pieces together cost him 110. How many pieces had he?

Let the number sought $= x$. By the question the purchaser paid for the different pieces of cloth in the following manner:

for the 1, 2, 3, 4, 5 x
he pays 2, 4, 6, 8, 10 .... $2x$ crowns.

It is therefore required to find the sum of the arithmetical progression $2 + 4 + 6 + 8 + 10 + \ldots\ldots 2x$, which consists of x terms, that we may deduce from it the price of all the pieces of cloth taken together. The rule which we have already given for this operation, requires us to add the last term and the first; the sum of which is $2x + 2$; if we multiply this sum by the number of terms x, the product will be $2xx + x$; if we lastly divide by the difference 2, the quotient will be $xx + x$, which is the sum of the progression; so that we have $xx + x = 110$; wherefore $xx = -x + 110$, and

$$x = -\frac{1}{2} + \sqrt{\frac{1}{4} + 110} = -\frac{1}{2} + \frac{21}{2} = 10.$$

Answer. The number of pieces of cloth is 10.

569. *Question* VII. A person bought several pieces of cloth, for 180 crowns. If he had received for the same sum 3 pieces more, he would have paid 3 crowns less for each piece; How many pieces did he buy?

Let us make the number sought $= x$; then each piece will have cost him $\frac{180}{x}$ crowns. Now, if the purchaser had had $x + 3$ pieces for 180 crowns, each piece would have cost $\frac{180}{x \times 3}$ crowns; and, since this price is less than the real price by three crowns, we have this equation,

$$\frac{180}{x+3} = \frac{180}{x} - 3.$$

Multiplying by x, we have $\frac{180\,x}{x+3} = 180 - 3\,x$; dividing by 3, we have $\frac{60\,x}{x+3} = 60 - x$; multiplying by $x + 3$, we have

$$60\,x = 180 + 57\,x - x\,x;$$

adding $x\,x$, we shall have $x\,x + 60\,x = 10 + 57\,x$; subtracting $60\,x$, we shall have $x\,x = -3\,x + 180$.

The rule, consequently, gives

$$x = -\frac{3}{2} + \sqrt{\frac{9}{4} + 180}, \text{ or } x = -\frac{3}{2} + \frac{27}{2} = 12.$$

Answer. He bought for 180 crowns 12 pieces of cloth at 15 crowns the piece, and if he had got 3 pieces more, namely, 15 pieces for 180 crowns, each piece would have cost only 12 crowns, that is to say, 3 crowns less.

570. *Question* VIII. Two merchants enter into partnership with a stock of 100 crowns; one leaves his money in the partnership for three months, the other leaves his for two months, and each takes out 99 crowns of capital and profit. What proportion of the stock did each furnish?

Suppose the first partner contributed x crowns, the other will have contributed $100 - x$. Now, the former receiving 99 crowns, his profit is $99 - x$, which he has gained in three months with the principal x; and since the second receives also 99 crowns, his profit is $x - 1$, which he has gained in two months with the principal $100 - x$; it is evident also, that the profit of this second partner would have been $\frac{3\,x - 3}{2}$, if he had remained three months in the partnership. Now, as the profits gained in the same time are in proportion to the principals, we have the following proportion,

$$x : 99 - x = 100 - x : \frac{3\,x - 3}{2}.$$

The equality of the product of the extremes to that of the means, gives the equation

$$\frac{3xx - 3x}{2} = 9900 - 199x + xx.$$

Multiplying by 2, we have

$$3xx - 3x = 19800 - 398x + 2xx;$$

subtracting $2xx$, we have $xx - 3x = 19800 - 398x$; adding $3x$, we have $xx = 19800 - 395x$.

Wherefore by the rule,

$$x = -\frac{395}{2} + \sqrt{\frac{156025}{4} + \frac{79200}{4}} = -\frac{395}{2} + \frac{485}{2} = \frac{90}{2} = 45.$$

Answer. The first partner contributed 45 crowns, and the other 55 crowns. The first, having gained 54 crowns in three months, would have gained in one month 18 crowns; and the second having gained 44 crowns in two months, would have gained 22 crowns in one month: now these profits agree; for, if with 45 crowns 18 crowns are gained in one month, 22 crowns will be gained in the same time with 55 crowns.

571. *Question* IX. Two girls carry 100 eggs to market; one had more than the other, and yet the sum which each received for them was the same. The first says to the second, if I had had your eggs, I should have received 15 sous. The other answers, if I had had yours, I should have received $6\frac{2}{3}$ sous. How many eggs did each carry to market?

Suppose the first had x eggs; then the second must have had $100 - x$.

Since therefore the former would have sold $100 - x$ eggs for 15 sous, we have the following proportion;

$$100 - x : 15 = x \ldots . \text{ to } \frac{15x}{100 - x} \text{ sous.}$$

Also, since the second would have sold x eggs for $6\frac{2}{3}$ sous, we find how much she got for $100 - x$ eggs, by saying

$$x : \frac{20}{3} = 100 - x \ldots . \text{ to } \frac{2000 - 20x}{3x}.$$

Now each of the girls received the same sum; we have consequently the equation,

$$\frac{15x}{100-x} = \frac{2000-20x}{3x},$$

which becomes this,

$$25xx = 200000 - 4000x;$$

and lastly this,

$$xx = -160x + 8000;$$

whence we obtain

$$x = -80 + \sqrt{6400+8000} = -80 + 120 = 40.$$

Answer. The first girl had 40 eggs, the second had 60, and each received 10 sous.

572. *Question* X. Two merchants sell each a certain quantity of stuff; the second sells 3 ells more than the first, and they received together 35 crowns. The first says to the second, I should have got 24 crowns for your stuff; the other answers, and I should have got for yours 12 crowns and a half. How many ells had each?

Suppose the first had x ells; then the second must have had $x+3$ ells. Now, since the first would have sold $x+3$ ells for 24 crowns, he must have received $\frac{24x}{x+3}$ crowns for his x ells. And with regard to the second, since he would have sold x ells for $12\frac{1}{2}$ crowns, he must have sold his $x+3$ ells for $\frac{25x-75}{2x}$; so that the whole sum they received was

$$\frac{24x}{x+3} + \frac{25x+75}{2x} = 35 \text{ crowns.}$$

This equation becomes $xx = 20x - 75$, whence we have

$$x = 10 \pm \sqrt{100-75} = 10 \pm 5.$$

Answer. The question admits of two solutions; according to the first, the first merchant had 15 ells, and the second had 18; and since the former would have sold 18 ells for 24 crowns, he must have sold his 15 ells for 20 crowns; the second, who would have sold 15 ells for 12 crowns and a half, must have sold his 18 ells for 15 crowns; so that they actually received 35 crowns for their commodity.

According to the second solution, the first merchant had 5 ells, and the other 8 ells; so that, since the first would have sold 8 ells for 24 crowns, he must have received 15 crowns for his 5 ells; and since the second would have sold 5 ells for 12 crowns and a half, his 8 ells must have produced him 20 crowns. The sum is, as before, 35 crowns.

CHAPTER VII.

Of the Nature of Equations of the Second Degree.

573. What we have already said sufficiently shows, that equations of the second degree admit of two solutions; and this property ought to be examined in every point of view, because the nature of equations of a higher degree will be very much illustrated by such an examination. We shall therefore retrace, with more attention, the reasons which render an equation of the second degree capable of a double solution; since they undoubtedly will exhibit an essential property of those equations.

574. We have already seen, it is true, that this double solution arises from the circumstance that the square root of any number may be taken either positively, or negatively; however, as this principle will not easily apply to equations of higher degrees, it may be proper to illustrate it by a distinct analysis. Taking, for an example, the quadratic equation, $xx = 12x - 35$, we shall give a new reason for this equation being resolvable in two ways, by admitting for x the values 5 and 7, both of which satisfy the terms of the equation.

575. For this purpose it is most convenient to begin with transposing the terms of the equation, so that one of the sides may become 0; this equation consequently takes the form

$$xx - 12x + 35 = 0;$$

and it is now required to find a number such, that if we substitute it for x, the quantity $xx - 12x + 35$ may be really equal to nothing; after this, we shall have to show how this may be done in two ways.

576. Now, the whole of this consists in showing clearly, that *a quantity of the form* $xx - 12x + 35$ *may be considered as the product of two factors;* thus, in fact, the quantity of which we speak

is composed of the two factors $(x - 5) \times (x - 7)$. For, since this quantity must become 0, we must also have the product

$$(x - 5) \times (x - 7) = 0;$$

but *a product, of whatever number of factors it is composed, becomes* $= 0$, *only when one of those factors is reduced to* 0; this is a fundamental principle to which we must pay particular attention, especially when equations of several degrees are treated of.

577. It is therefore easily understood, that *the product*

$$(x - 5) \times (x - 7)$$

may become 0 *in two ways; one, when the first factor* $x - 5 = 0$; *the other, when the second factor,* $x - 7 = 0$. *In the first case* $x = 5$, *in the other,* $x = 7$. The reason is, therefore, very evident, why such an equation $x x - 12 x + 35 = 0$, admits of two solutions, that is to say, why we can assign two values of x, both of which equally satisfy the terms of the equation. This fundamental principle consists in this, that the quantity $x x - 12 x + 35$ may be represented by the product of two factors.

578. The same circumstances are found in all equations of the second degree. For, after having brought all the terms to one side, we always find an equation of the following form $x x - a x + b = 0$, and this formula may be always considered as the product of two factors, which we shall represent by $(x - p) \times (x - q)$, without concerning ourselves what numbers the letters p and q represent. Now, as this product must be $= 0$, from the nature of our equation it is evident that this may happen in two ways; in the first place, when $x = p$; and in the second place, when $x = q$; and these are the two values of x which satisfy the terms of the equation.

579. Let us now consider the nature of these two factors, in order that the multiplication of the one by the other may exactly produce $x x - a x + b$. By actually multiplying them, we get $x x - (p + q) x + p q$; now this quantity must be the same as $x x - a x + b$, wherefore we have evidently $p + q = a$, and $p q = b$. So that we have deduced this very remarkable property, that *in every equation of the form* $x x - a x + b = 0$, *the two values of* x *are such, that their sum is equal to* a, *and their product equal to* b; *whence it follows that, if we know one of the values, the other also is easily found.*

580. We have considered the case in which the two values of x are positive, and which requires the second term of the equation to have the sign $-$, and the third term to have the sign $+$. Let us also consider the cases in which either one or both values of x become negative. The first takes place when the two factors of the equation give a product of this form $(x-p) \times (x+q)$; for then the two values of x are $x = p$, and $x = -q$; the equation itself becomes $x x + (q-p) x - p q = 0$; the second term has the sign $+$, when q is greater than p, and the sign $-$, when q is less than p; lastly, the third term is always negative.

The second case, in which both values of x are negative, occurs, when the two factors are $(x+p) \times (x+q)$; for we shall then have $x = -p$ and $x = -q$; the equation itself becomes

$$x x + (p+q) x + p q = 0,$$

in which both the second and third terms are affected by the sign $+$.

581. The signs of the second and the third term consequently show us the nature of the roots of any equation of the second degree. Let the equation be $x x \ldots . a x \ldots . b = 0$, if the second and third terms have the sign $+$, the two values of x are both negative; if the second term has the sign $-$, and the third term has $+$, both values are positive; lastly, if the third term also has the sign $-$, one of the values in question is positive. But in all cases, whatever, the second term contains the sum of the two values, and the third term contains their product.

582. After what has been said, it will be very easy to form equations of the second degree containing any two given values. Let there be required, for example, an equation such, that one of the values of x may be 7, and the other -3. We first form the simple equations $x = 7$ and $x = -3$; thence these $x - 7 = 0$ and $x + 3 = 0$, which gives us, in this manner, the factors of the equation required, which consequently becomes $x x - 4 x - 21 = 0$. Applying here, also, the above rule, we find the two given values of x; for if $x x = 4 x + 21$, we have $x = 2 \pm \sqrt{25} = 2 \pm 5$, that is to say, $x = 7$, or $x = -3$.

583. The values of x may also happen to be equal. Let there be sought, for example, an equation, in which both values may be $= 5$. The two factors will be $(x-5) \times (x-5)$, and the equations sought will be $x x - 10 x + 25 = 0$. In this equation, x appears to have only one value; but it is because x is twice found

$= 5$, as the common method of resolution shows; for we have $xx = 10x - 25$; wherefore $x = 5 \pm \sqrt{0} = 5 \pm 0$, that is to say, x is in two ways $= 5$.

584. A very remarkable case, in which both values of x become imaginary, or impossible, sometimes occurs; and it is then wholly impossible to assign any value for x, that would satisfy the terms of the equation. Let it be proposed, for example, to divide the number 10 into two parts, such, that their product may be 30. If we call one of those parts x, the other will be $= 10 - x$, and their product will be $10x - xx = 30$; wherefore $xx = 10x - 30$, and $x = 5 \pm \sqrt{-5}$, which being *an imaginary number, shows that the question is impossible.*

585. It is very important, therefore, to discover some sign, by means of which he may immediately know, whether an equation of the second degree is possible or not.

Let us resume the general equation $ax - xx + b = 0$. We shall have

$$xx = ax - b, \text{ and } x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - b}.$$

This shows, that if b is greater than $\frac{1}{4}aa$, or $4b$ greater than aa, the two vlaues of x are always imaginary, since it would be required to extract the square root of a negative quantity; on the contrary, if b is less than $\frac{1}{4}aa$, or even less than 0, that is to say, is a negative number, both values will be possible or real. But whether they be real or imaginary, it is no less true, that they are still expressible, and always have this property, that their sum is $= a$, and their product $= b$. In the equation $xx - 6x + 10 = 0$, for example, the sum of the two values of x must be $= 6$, and the product of these two values must be $= 10$; now we find, I. $x = 3 + \sqrt{-1}$, and II. $x = 3 - \sqrt{-1}$, quantities whose sum $= 6$, and the product $= 10$.

586. The expression, which we have just found, may be represented in a manner more general, and so as to be applied to equations of this form, $fxx \pm gx + h = 0$; for this equation gives

$$xx = \pm \frac{gx}{f} - \frac{h}{f},$$

and

$$x = \pm \frac{g}{2f} \pm \sqrt{\frac{gg}{4ff} - \frac{h}{f}},$$

or $$x = \frac{\pm g \pm \sqrt{gg - 4fh}}{2f};$$

whence we conclude that the two values are imaginary, and consequently the equation impossible, when $4fh$ is greater than gg; that is to say, when, in the equation $fxx - gx - h = 0$, four times the product of the first and the last term exceeds the square of the second term: for the product of the first and the last term, taken four times, is $4fhxx$, and the square of the middle term is $ggxx$; now, if $4fhxx$ is greater than $ggxx$, $4fh$ is also greater than gg, and in that case, the equation is evidently impossible. In all other cases the equation is possible, and two real values of x may be assigned. It is true they are often irrational; but we have already seen, that, in such cases, we may always find them by approximation; whereas no approximations can take place with regard to imaginary expressions, such as $\sqrt{-5}$; for 100 is as far from being the value of that root, as 1, or any other number.

587. We have further to observe, that *any quantity of the second degree,* $xx \pm ax \pm b$, *must always be resolvable into two factors,* such as $(x \pm p) \times (x \pm q)$. For, if we took three factors, such as these, we should come to a quantity of the third degree, and taking only one such factor, we should not exceed the first degree.

It is therefore certain that *every equation of the second degree necessarily contains two values of* x, *and that it can neither have more nor less.*

588. We have already seen, that when the two factors are found, the two values of x are also known, since each factor gives one of those values, when it is supposed to be $=0$. The converse also is true, *viz.* that when we have found one value of x, we know also one of the factors of the equation; for if $x = p$ represents one of the values of x, in any equation of the second degree, $x - p$ is one of the factors of that equation; that is to say, all the terms having been brought to one side, the equation is divisible by $x - p$; and further, the quotient expresses the other factor.

489. In order to illustrate what we have now said, let there be given the equation $xx + 4x - 21 = 0$, in which we know that $x = 3$ is one of the values of x, because $\overline{3+3} + 4\,\overline{+3} - 21 = 0$; this shows, that $x - 3$ is one of the factors of the equation, or that $xx + 4x - 21$ is divisible by $x - 3$, which the actual division proves.

$$
\begin{array}{r}
x - 3)\ xx + 4x - 21\ (x + 7 \\
xx - 3x \qquad \\
\hline
7x - 21 \\
7x - 21 \\
\hline
0.
\end{array}
$$

So that the other factor is $x + 7$, and our equation is represented by the product $(x - 3) \times (x + 7) = 0$; whence the two values of x immediately follow, the first factor giving $x = 3$, and the other $x = -7$.

QUESTIONS FOR PRACTICE.

Fractions.

SECTION I.—CHAPTER 9.

1. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to a common denominator.

Ans. $\frac{2cx}{ac}$ and $\frac{ab}{ac}$.

2. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to a common denominator.

Ans. $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$.

3. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d to fractions having a common denominator.

Ans. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

4. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a+\frac{2x}{a}$ to a common denominator.

Ans. $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2+24x}{12a}$.

5. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$, and $\frac{x^2+a^2}{x+a}$ to a common denominator.

Ans. $\frac{3x+3a}{6x+6a}$, $\frac{2a^2x+2a^3}{6x+6a}$, and $\frac{6x^2+6a^2}{6x+6a}$.

6. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$, and $\frac{d}{a}$ to a common denominator.

Ans. $\frac{2a^2b}{4a^4}$, $\frac{2a^3c}{4a^4}$, and $\frac{4a^3d}{4a^4}$.

SECTION I.—CHAPTER 10.

7. Required the product of $\frac{x}{6}$ and $\frac{2x}{9}$. *Ans.* $\frac{x^2}{27}$.

8. Required the product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$. *Ans.* $\frac{4x^3}{21}$.

9. Required the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$. *Ans.* $\frac{x^2+ax}{a^2+ac}$.

10. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. *Ans.* $\frac{9ax}{2b}$.

11. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. *Ans.* $\frac{3x^3}{5a}$.

12. Required the product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$. *Ans.* $9ax$.

13. Required the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$. *Ans.* $\frac{ab+bx}{x}$.

14. Required the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

Ans. $\frac{x^4-b^4}{b^2c+bc^2}$.

15. Required the product of x, $\frac{x+1}{a}$, and $\frac{x-1}{a+b}$.

Ans. $\frac{x^3-x}{a^2+ab}$.

16. Required the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$. *Ans.* $1\frac{1}{2}$.

17. Required the quotient of $\frac{2a}{b}$ divided by $\frac{4c}{d}$. *Ans.* $\frac{ad}{2bc}$.

18. Required the quotient of $\frac{x+a}{2x-2b}$ divided by $\frac{x+b}{5x+a}$.

Ans. $\frac{5x^2+6ax+a^2}{2x^2-2b^2}$.

19. Required the quotient of $\frac{2x^2}{a^3+x^3}$ divided by $\frac{x}{x+a}$.

Ans. $\frac{2x}{x^2-ax+a^2}$.

20. Required the quotient of $\frac{7x}{5}$ divided by $\frac{12}{13}$. *Ans.* $\frac{91x}{60}$.

21. Required the quotient of $\frac{4x^2}{7}$ divided by $5x$. *Ans.* $\frac{4x}{35}$.

22. Required the quotient of $\frac{x+1}{5}$ divided by $\frac{2x}{3}$. *Ans.* $\frac{x+1}{4x}$.

23. Required the quotient of $\frac{x-b}{8cd}$ divided by $\frac{3cx}{4d}$. *Ans.* $\frac{x-b}{6c^2x}$.

24. Required the quotient of $\frac{x^4-b^4}{x^2-2bx+b^2}$ divided by $\frac{x^2+bx}{x-b}$. *Ans.* $x+\frac{b^2}{x}$.

Infinite Series.

SECTION II.—CHAPTER 5.

25. Resolve $\frac{ax}{a-x}$ into an infinite series.

Ans. $x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}$, &c.

26. Resolve $\frac{b}{a+x}$ into an infinite series.

Ans. $\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+$ &c.

Or resolved into factors,

$\frac{b}{a}\times(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$ &c.)

27. Resolve $\frac{a^2}{x+b}$ into an infinite series.

Ans. $\frac{a^2}{x}\times(1-\frac{b}{x}+\frac{b^2}{x^2}-\frac{b^3}{x^3}+$ &c.)

28. Resolve $\frac{1+x}{1-x}$ into an infinite series.

Ans. $1+2x+2x^2+2x^3+2x^4$, &c.

29. Resolve $\frac{a^2}{(a+x)^2}$ into an infinite series.

Ans. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}$, &c.

Surds or Irrational Numbers.

SECTION I.—CHAPTERS 12, 19; AND SECTION II.—CHAPTER 8, &c.

30. Reduce 6 to the form of $\sqrt{5}$. *Ans.* $\sqrt{36}$.

31. Reduce $a+b$ to the form of $\sqrt{bc}$. *Ans.* $\sqrt{aa+2ab+bb}$.

32. Reduce $\frac{a}{b\sqrt{c}}$ to the form of $\sqrt{d}$. *Ans.* $\sqrt{\frac{aa}{bbc}}$.

33. Reduce a^2 and $b^{\frac{3}{2}}$ to the common exponent $\frac{1}{3}$.

Ans. $\left(a^6\right)^{\frac{1}{3}}$, and $\left(b^{\frac{9}{2}}\right)^{\frac{1}{3}}$.

34. Reduce $\sqrt{48}$ to its simplest form. *Ans.* $4\sqrt{3}$.

35. Reduce $\sqrt{a^3x-a^2x^2}$ to its simplest form.

Ans. $a\sqrt{ax-xx}$.

36. Reduce $\sqrt[3]{\frac{27a^4b^3}{8b-8a}}$ to its simplest form.

Ans. $\frac{3ab}{2}\sqrt[3]{\frac{a}{b-a}}$.

37. Add $\sqrt{6}$ to $2\sqrt{6}$; and $\sqrt{8}$ to $\sqrt{50}$. *Ans.* $3\sqrt{6}$; and $\sqrt{98}$.

38. Add $\sqrt{4a}$ and $\sqrt[4]{a^6}$ together. *Ans.* $(a+2)\sqrt{a}$.

39. Add $\left(\frac{b}{c}\right)^{\frac{1}{2}}$ and $\left(\frac{c}{b}\right)^{\frac{3}{2}}$ together. *Ans.* $\frac{bb+cc}{b\sqrt{bc}}$.

40. Subtract $\sqrt{4a}$ from $\sqrt[4]{a^6}$. *Ans.* $(a-2)\sqrt{a}$.

41. Subtract $\left(\frac{c}{b}\right)^{\frac{3}{2}}$ from $\left(\frac{b}{c}\right)^{\frac{1}{2}}$. *Ans.* $\frac{bb-cc}{b}\sqrt{\frac{1}{bc}}$.

42. Multiply $\sqrt{\frac{2ab}{3c}}$ by $\sqrt{\frac{9ad}{2b}}$. *Ans.* $\sqrt{\frac{3a^2d}{c}}$.

43. Multiply $\sqrt{d}$ by $\sqrt[3]{ab}$. *Ans.* $\sqrt[6]{a^2b^2d^3}$.

44. Multiply $\sqrt{4a-3x}$ by $2a$. *Ans.* $\sqrt{16a^3-12a^2x}$.

45. Multiply $\frac{a}{2b}\sqrt{a-x}$ by $(c-d)\sqrt{ax}$.

Ans. $\frac{ac-ad}{2b}\sqrt{a^2x-ax^2}$.

46. Multiply $\sqrt{a} - \sqrt{b} - \sqrt{3}$ by $\sqrt{a} + \sqrt{b} - \sqrt{3}$.

Ans. $\sqrt{a^2 - b} + \sqrt{3}$.

47. Divide $a^{\frac{2}{3}}$ by $a^{\frac{1}{4}}$; and $a^{\frac{1}{n}}$ by $a^{\frac{1}{m}}$. *Ans.* $a^{\frac{5}{12}}$ and $a^{\frac{m-n}{m\,n}}$.

48. Divide $\frac{a\,c - a\,d}{2\,b} \sqrt{a^2\,x - a\,x^2}$ by $\frac{a}{2\,b} \sqrt{a - x}$.

Ans. $(c - d) \sqrt{a\,x}$.

49. Divide $a^2 - a\,d - b + d\sqrt{b}$ by $a - \sqrt{b}$.

Ans. $a + \sqrt{b} - d$.

50. What is the cube of $\sqrt{2}$? *Ans.* $\sqrt{8}$.

51. What is the square of $3\sqrt[3]{b\,c^2}$? *Ans.* $9\,c\sqrt[3]{b^2\,c}$.

52. What is the fourth power of $\frac{a}{2\,b}\sqrt{\frac{2\,a}{c - b}}$?

Ans. $\frac{a^6}{4\,b^4\,(c^2 - 2\,bc + b^2)}$.

53. What is the square of $3 + \sqrt{5}$? *Ans.* $14 + 6\sqrt{5}$.

54. What is the square root of a^3? *Ans.* $a^{\frac{3}{2}}$, or $\sqrt{a^3}$.

55. What is the cube root of $a\,b^4$? *Ans.* $a\,b\,b^{\frac{1}{3}}$; or $\sqrt[3]{a\,b\,b}$.

56. What is the cube root of $\sqrt{a^2 - x^2}$? *Ans.* $\sqrt[6]{a^2 - x^2}$.

57. What is the cube root of $a^2 - \sqrt{a\,x - x^2}$?

Ans. $\sqrt[3]{a^2 - \sqrt{a\,x - x^2}}$.

58. What multiplier will render $a + \sqrt{3}$ rational?

Ans. $a - \sqrt{3}$.

59. What multiplier will render $\sqrt{a} - \sqrt{b}$ rational?

Ans. $\sqrt{a} + \sqrt{b}$.

60. What multiplier will render the denominator of the fraction $\frac{\sqrt{6}}{\sqrt{7} + \sqrt{3}}$ rational? *Ans.* $\sqrt{7} - \sqrt{3}$.

SECTION II.—CHAPTER 12.

61. Resolve $\sqrt{a^2 + x^2}$ into an infinite series.

Ans. $a + \frac{x^2}{2\,a} - \frac{x^4}{8\,a^3} + \frac{x^6}{16\,a^5} - \frac{5\,x^8}{828\,a^7}$, &c.

62. Resolve $\sqrt{1+1}$ into an infinite series.

$$Ans.\ 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}, \text{ \&c.}$$

63. Resolve $\sqrt{a^2 - x^2}$ into an infinite series.

$$Ans.\ a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}, \text{ \&c.}$$

64. Resolve $\sqrt[3]{1 - x^3}$ into an infinite series.

$$Ans.\ 1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81}, \text{ \&c.}$$

65. Resolve $\sqrt{r^2 - x^2}$ into an infinite series.

$$Ans.\ r - \frac{x^2}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7}, \text{ \&c.}$$

66. Resolve $\frac{1}{\sqrt{a^2 - x^2}}$ into an infinite series.

$$Ans.\ \frac{1}{a} + \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} + \frac{15x^6}{48a^7}, \text{ \&c.}$$

67. Resolve $(a^2 - x^2)^{\frac{1}{5}}$ into an infinite series.

$$Ans.\ a^{\frac{2}{5}} \times (1 - \frac{x^2}{5a^2} - \frac{2x^4}{25a^4} - \frac{6x^6}{125a^6} - \text{ \&c.}$$

68. Resolve $\sqrt{\frac{a^2 + x^2}{a^2 - x^2}}$ into an infinite series.

$$Ans.\ 1 + \frac{x^2}{a^2} + \frac{x^4}{2a^4} + \frac{x^6}{2a^6}, \text{ \&c.}$$

69. Resolve $\sqrt[3]{\frac{a^3 + x^3}{(a^2 + x^2)^3}}$ into an infinite series.

$$Ans.\ \frac{1}{a\sqrt[3]{a}} \times (1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6} + \text{ \&c.}$$

Summation of Arithmetical Progressions.

SECTION III.—CHAPTER 4.

70. Required the sum of an increasing arithmetical progression, having 3 for its first term, 2 for the common difference, and the number of terms 20. *Ans.* 440.

71. Required the sum of a decreasing arithmetical progression, having 10 for its first term, $\frac{1}{3}$ for the common difference, and the number of terms 21. *Ans.* 140.

72. Required the number of all the strokes of a clock in twelve hours, that is, a complete revolution of the index. *Ans*. 78.

73. The clocks of Italy go on to 24 hours; how many strokes do they strike in a complete revolution of the index? *Ans*. 300.

74. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other, how far will a person travel who shall bring them one by one to a basket, which is placed one yard from the first stone.

Ans. 5 *miles* and 1300 *yards*.

The greatest Common Divisor.

SECTION III. CHAPTER 6.—SECTION I. CHAPTER 8.

75. Reduce $\frac{c\,x + x^2}{c\,a^2 + a^2\,x}$ to its lowest terms. *Ans*. $\frac{x}{a^2}$.

76. Reduce $\frac{x^3 - b^2\,x}{x^2 + 2\,b\,x + b^2}$ to its lowest terms. *Ans*. $\frac{x^2 - b\,x}{x + b}$.

77. Reduce $\frac{x^4 - b^4}{x^5 - b^2\,x^3}$ to its lowest terms. *Ans*. $\frac{x^2 + b^2}{x^3}$.

78. Reduce $\frac{x^2 - y^2}{x^4 - y^4}$ to its lowest terms. *Ans*. $\frac{1}{x^2 + y^2}$.

79. Reduce $\frac{a^4 - x^4}{a^3 - a^2\,x + a\,x^2 - x^3}$ to its lowest terms.

Ans. $\frac{a + x}{1}$.

80. Reduce $\frac{5\,a^5 + 10\,a^4\,x + 5\,a^3\,x^2}{a^3\,x + 2\,a^2\,x^2 + 2\,a\,x^3 + x^4}$ to its lowest terms.

Ans. $\frac{5\,a^4 + 5\,a^3\,x}{a^2\,x + a\,x^2 + x^3}$.

Summation of Geometrical Progressions.

SECTION III.—CHAPTER 10.

81. A SERVANT agreed with a master to serve him eleven years without any other reward for his service than the produce of one wheat corn for the first year; and that product to be sown the second year, and so on from year to year till the end of the time, allowing the increase to be only in a tenfold proportion. What was the sum of the whole produce? *Ans*. 11111111110 *wheat corns*.

N. B. It is further required to reduce this number of corns to the proper measures of capacity, and then by supposing an average price of wheat, to compute the value of the corns in money.

82. A servant agreed with a gentleman to serve him twelve months, provided he would give him a farthing for his first month's service, a penny for the second, and 4*d.* for the third, &c. What did his wages amount to? *Ans.* 5825*l.* 8*s.* $5\frac{1}{4}$*d.*

83. *Sessa*, an *Indian*, having invented the game of chess, showed it to his prince, who was so delighted with it, that he promised him any reward he should ask; upon which Sessa requested that he might be allowed one grain of wheat for the first square on the chess-board, two for the second, and so on, doubling continually, to 64, the whole number of squares; now supposing a pint to contain 7680 of those grains, and one quarter to be worth 1*l.* 7*s.* 6*d.*, it is required to compute the value of the whole sum of grains.

Ans. £64481488296.

Simple Equations.

SECTION IV.—CHAPTER 2.

84. If $x - 4 + 6 = 8$, then will $x = 6$.

85. If $4x - 8 = 3x + 20$, then will $x = 28$.

86. If $ax = ab - a$, then will $x = b - 1$.

87. If $2x + 4 = 16$, then will $x = 6$.

88. If $ax + 2ba = 3c^2$, then will $x = \frac{3c^2}{a} - 2b$.

89. If $\frac{x}{2} = 5 + 3$, then will $x = 16$.

90. If $\frac{2x}{3} - 2 = 6 + 4$, then will $x = 18$.

91. If $a - \frac{b}{x} = c$, then will $x = \frac{b}{a - c}$.

92. If $5x - 15 = 2x + 6$, then will $x = 7$.

93. If $40 - 6x - 16 = 120 - 14x$, then will $x = 12$.

94. If $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 10$, then will $x = 24$.

95. If $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x-19}{2}$, then will $x = 23\frac{1}{4}$.

96. If $\sqrt{\frac{2}{3}x + 5} = 7$, then will $x = 6$.

97. If $x + \sqrt{a^2 + x^2} = \frac{2\,a^2}{\sqrt{a^2 + x^2}}$, then will $x = a\sqrt{\tfrac{1}{3}}$.

98. If $3\,a\,x + \frac{a}{2} - 3 = b\,x - a$, then will $x = \frac{6 - 3\,a}{6\,a - 2\,b}$.

99. If $\sqrt{12 + x} = 2 + \sqrt{x}$, then will $x = 4$.

100. If $y + \sqrt{a^2 + y^2} = \frac{2\,a^2}{(a^2 + y^2)^{\frac{1}{2}}}$, then will $y = \frac{1}{3}\,a\sqrt{3}$.

101. If $\frac{y+1}{2} + \frac{y+2}{3} = 16 - \frac{y+3}{4}$, then will $y = 13$.

102. If $\sqrt{x}, + \sqrt{a + x} = \frac{2\,a}{\sqrt{a + x}}$, then will $x = \frac{a}{3}$.

103. If $\sqrt{a^2 + x^2} = \sqrt[4]{b^4 + x^4}$, then will $x = \sqrt{\frac{b^4 - a^4}{2\,a^2}}$.

104. If $x = \sqrt{a^2 + x\sqrt{b^2 + x^2}} - a$, then will $x = \frac{b^2}{4a} - a$.

105. If $\frac{128}{3\,x - 4} = \frac{216}{5\,x - 6}$, then will $x = 12$.

106. If $\frac{42\,x}{x - 2} = \frac{35\,x}{x - 3}$, then will $x = 8$.

107. If $\frac{45}{2\,x + 3} = \frac{57}{4x - 5}$, then will $x = 6$.

108. If $\frac{x^2 - 12}{3} = \frac{x^2 - 4}{4}$, then will $x = 6$.

109. If $615\,x - 7\,x^2 = 48\,x$, then will $x = 9$.

SECTION IV.—CHAPTER 3.

110. To find a number, to which, if there be added a half, a third, and a fourth of itself, the sum will be 50. *Ans.* 24.

111. A person being asked what his age was, replied, that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age gives a product equal to his age. What was his age? *Ans.* 16.

112. The sum of 660*l.* was raised for a particular purpose by four persons, A, B, C, and D; B advanced twice as much as A; C as much as A and B together; and D as much as B and C. What did each contribute? *Ans.* 60*l.*, 120*l.*, 180*l.*, and 300*l.*

113. To find that number whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 12. *Ans.* 144.

114. What sum of money is that, whose $\frac{1}{3}$ part, $\frac{1}{4}$ part, and $\frac{1}{5}$ part added together, amount to 94 pounds? *Ans.* 120*l.*

115. In a mixture of copper, tin, and lead, one half of the whole — 16*lb.* was copper; $\frac{1}{3}$ of the whole — 13*lb.* tin; and $\frac{1}{4}$ of the whole + 4*lb.* lead. What quantity of each was there in the composition? *Ans.* 128*lb.* of copper, 84*lb.* of tin, and 76*lb.* of lead.

116. What number is that, whose $\frac{1}{3}$ part exceeds its $\frac{1}{5}$ by 72? *Ans.* 540.

117. To find two numbers in the proportion of 2 to 1, so that if 4 be added to each, the two sums shall be in the proportion of 3 to 2. *Ans.* 8 and 4.

118. There are two numbers such that $\frac{1}{2}$ of the greater added to $\frac{1}{3}$ of the less is 13, and if $\frac{1}{2}$ of the less be taken from $\frac{1}{3}$ of the greater, the remainder is nothing. What are the numbers? *Ans.* 18 and 12.

119. In the composition of a certain quantity of gunpowder $\frac{2}{3}$ of the whole plus 10 was *nitre*; $\frac{1}{6}$ of the whole minus $4\frac{1}{2}$ was *sulphur*, and the *charcoal* was $\frac{1}{7}$ of the nitre — 2. How many pounds of gunpowder were there? *Ans.* 69.

120. A person has a lease for 99 years; and being asked how much of it was already expired, answered, that two thirds of the time past was equal to four fifths of the time to come. Required the time past. *Ans.* 54 years.

121. It is required to divide the number 48 into two such parts, that the one part may be three times as much above 20 as the other wants of 20. *Ans.* 32 and 16.

122. A person rents 25 acres of land at 7 pounds 12 shillings per annum; this land consisting of two sorts, he rents the better sort at 8 shillings per acre, and the worse at 5. Required the number of acres of the better sort. *Ans.* 9.

123. A certain cistern, which would be filled in 12 minutes by two pipes running into it, would be filled in 20 minutes by one alone. Required, in what time it would be filled by the other alone. *Ans.* 30 minutes.

124. Required two numbers, whose sum may be s, and their proportion as a to b. *Ans.* $\frac{as}{a+b}$ and $\frac{bs}{a+b}$.

125. A privateer running at the rate of 10 miles an hour, discovers a ship 18 miles off making way at the rate of 8 miles an hour; it is demanded how many miles the ship can run before she will be overtaken? *Ans.* 72.

126. A gentleman distributing money among some poor people, found he wanted 10*s*. to be able to give 5*s*. to each; therefore he gives 4*s*. only, and finds that he has 5*s*. left. Required the number of shillings and of poor people.

Ans. 15 poor people, and 65 shillings.

127. There are two numbers whose sum is the sixth part of their product, and the greater is to the less as 3 to 2. Required those numbers. *Ans*. 15 and 10.

N. B. This question may be solved likewise by means of one unknown letter.

128. To find three numbers, such that the first, with half the other two, the second with one third of the other two, and the third with one fourth of the other two, may be equal to 34.

Ans. 26, 22, and 10.

129. To find a number consisting of three places, whose digits are in arithmetical progression; if this number be divided by the sum of its digits, the quotient will be 48; and if from the number be subtracted 198, the digits will be inverted. *Ans*. 432.

130. To find three numbers such, that $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall be equal to 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, equal to 38. *Ans*. 24, 60, 120.

131. To find three numbers such that the first with $\frac{1}{2}$ of the sum of the second and third shall be 120, the second with $\frac{1}{6}$ of the difference of the third and first shall be 70, and $\frac{1}{2}$ of the sum of the three numbers shall be 95. *Ans*. 50, 63, 75.

132. What is that fraction which will become equal to $\frac{1}{3}$, if an unit be added to the numerator; but on the contrary, if an unit be added to the denominator, it will be equal to $\frac{1}{4}$? *Ans*. $\frac{4}{15}$.

133. The dimensions of a certain rectangular floor are such, that if it had been 2 feet broader, and 3 feet longer, it would have been 64 square feet larger; but if it had been 3 feet broader and 2 feet longer, it would then have been 68 square feet larger. Required the length and breadth of the floor.

Ans. Length 14 feet, and breadth 10 feet.

134. A person found that upon beginning the study of his profession $\frac{1}{7}$ of his life hitherto had passed before he commenced his education, $\frac{1}{3}$ under a private teacher, and the same time at a public school, and four years at the university. What was his age?

Ans. 21 years.

135. To find a number such that whether it be divided into two or three equal parts the continued product of the parts shall be equal to the same quantity. *Ans.* $6\frac{3}{4}$.

136. There is a certain number consisting of two digits. The sum of these digits is 5, and if 9 be added to the number itself the digits will be inverted. What is the number? *Ans.* 23.

137. What number is that, to which if I add 20 and from $\frac{2}{3}$ of this sum I subtract 12, the remainder shall be 10? *Ans.* 13.

Quadratic Equations.

SECTION IV.—CHAPTER 5.

138. To find that number to which 20 being added, and from which 10 being subtracted, the square of the sum, added to twice the square of the remainder, shall be 17475. *Ans.* 75.

139. What two numbers are those which are to one another in the ratio of 3 to 5, and whose squares, added together, make 1666? *Ans.* 21 and 35.

140. The sum $2a$, and the sum of the squares $2b$, of two numbers being given; to find the numbers.

Ans. $a - \sqrt{b - a^2}$ and $a + \sqrt{b - a^2}$.

141. To divide the number 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

142. To find three such numbers, that the sum of the first and second multiplied into the third, may be equal to 63; and the sum of the second and third, multiplied into the first equal to 28; also, that the sum of the first and third, multiplied into the second, may be equal to 55. *Ans.* 2, 5, 9.

143. What two numbers are those, whose sum is to the greater as 11 to 7; the difference of their squares being 132?

Ans. 14 and 8.