FIRST LESSONS of art-

IN

ALGEBRA;

BEING AN

Easy Introduction to that Science.

DESIGNED FOR THE USE OF

ACADEMIES AND COMMON SCHOOLS.

BY

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of arranging them in such a manner as may render the introduction to the science easy. If there be any peculiarity in this work, it is its simplicity. I have endeavored to make it as plain and intelligible as possible. There is little danger

Lottie Kelsey

PREFACE.

THIS treatise is especially intended for the use of beginners. I have long wished that Algebra might be introduced into common schools, as a standard branch of education; and there seems to be no good reason why the study of this most interesting and useful science should be confined to the higher seminaries of learning. The upper classes, at least, in common schools, might be profitably instructed in its elements, without neglecting any of those branches to which they usually attend.

This work pretends to no original investigations, no new discoveries. My labor has been the very humble one of selecting such materials as belong to the elements of Algebra, and of arranging them in such a manner as may render the introduction to the science easy. If there be any peculiarity in this work, it is its simplicity. I have endeavored to make it as plain and intelligible as possible. There is little danger that the student will find the beginning of any art or science too easy; and, in Algebra, he is required to learn a peculiar language, to determine new principles, and to accustom himself to an abstract mode of reasoning, with which he has been little acquainted. Let the explanations, therefore, be as full and diffuse as they may, he will still find difficulties enough to exercise his mind. I have aimed to prepare a work, which any boy of twelve years, who is thoroughly acquainted with the fundamental rules of Arithmetic, can understand, even without the aid of a teacher.

The following are the leading principles which I have observed, in preparing this treatise:—

To introduce only such parts of the science, as properly belong to an elementary work;

To adhere strictly to a methodical arrangement, that can be easily understood and remembered;

Never to anticipate principles, so as to make a clear understanding of the subject under consideration, depend upon some explanation which is to follow;

To introduce every new principle distinctly by itself, that the learner may encounter but one difficulty at a time;

To deduce the rules, generally, from practical exercises, and to state them distinctly and in form;

To give a great variety of questions for practice under each rule;

To solve or fully explain all questions which involve a new principle, or the new application of a principle already explained;

To show the reason of every step, without perplexing the learner with abstruse demonstrations;

To illustrate the nature of algebraic calculations, and their correctness, by a frequent reference to numbers;

And, finally, to advance from simple to difficult problems in such a manner as may fully exercise the powers of the learner without discouraging him.

As this little book professes to be merely an introduction to more full and scientifical treatises upon Algebra, it was not my original design to extend it beyond Equations of the First Degree. The subsequent chapters on Evolution and Equations of the Second Degree, have been added with a particular reference to schools for young ladies. It is presumed that the work, in its present form, contains as much of Algebra as this class of learners will, in general, find time to study.

E. Bailey.

Boston, *July*, 1833.

PUBLISHERS' ADVERTISEMENT.

IN presenting to the public this Revised Edition of Bailey's First Lessons in Algebra, the publishers beg leave to state, that the labor of revision has been undertaken by a daughter of the author. The task has thus, fortunately, fallen into the hands of one whose regard for the memory of a father, no less than a long experience in teaching, especially in this branch of study, has rendered her peculiarly well qualified for the undertaking. No pains have been spared to adapt the work to the present demand for an elementary text book in Algebra. Every page has been carefully and critically examined by Mr. S. S. Greene, Professor of Mathematics and Civil Engineering, in Brown University, to whose valuable suggestions, the publishers would acknowledge themselves indebted, not only for the improved arrangement of the matter, but for the lucid and elementary treatment of many of the subjects discussed.

Bailey's Algebra was one of the first and most successful attempts to adapt the subject to the young beginner. While the general spirit and aim of the author has been preserved in this edition, the book has been entirely re-written, and enlarged by the addition of copious examples, and by the discussion of several topics not included in the original plan. Some of the distinctive features of the revised edition, are,—

1. An Introductory Chapter, designed especially for beginners.

2. A full development of the subject of Factors, Multiples, and Divisors.

3. A complete discussion of the Square and Cube Roots.

4. A chapter on Arithmetical and Geometrical Progression.

5. A chapter on Surds, and Equations involving them.

6. A distinct statement of the principles developed, arranged, and numbered consecutively, for convenience of reference.

Hoping that this new edition will prove as valuable a text book to the present generation, as the former edition has to the past, we ask for it the attention of Teachers, School Committees, and the friends of Education generally.

HICKLING, SWAN & BREWER.

BOSTON, *October*, 1859.

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CHAPTER I.

INTRODUCTORY EXERCISES.

SECTION I.

EQUALITY AND INEQUALITY.

Commenced - Oct 17th 1861.
Recommenced - April 7, 1862.

THOMAS had 10 cents in his pocket, and Henry had 5 in each hand. Had they an equal or an unequal number? Thomas spends 4 of his, and Henry 2 of his. Have they now an equal or an unequal number? u Then,

1. *Two numbers may be either* EQUAL *or* UNEQUAL.

Are 10 and 10 equal or unequal? e Are

7 and $\frac{21}{3}$? e
$9\frac{3}{4}$ and 10? u
$3\frac{1}{4}$ and 3.25? e
$3\frac{1}{5}$ and $\frac{32}{10}$? e
.0001 and .00001? u
3.001 and 3.00100? e
0 and .000001?
$7\frac{1}{2}$ and $7\frac{22}{44}$?

Is 12 greater or less than 12.001?

$6\frac{1}{2}$ than $6\frac{3}{4}$?
.0001 than .00001? g
0 than .000001? l
$\frac{3}{4}$ than $\frac{5}{8}$? g
5,75 than $5\frac{2}{3}$? g
10.5 than 10.49? g

2. *To show that two numbers are equal, place between them two parallel lines* (=); *to show that two numbers are unequal, place between them two lines, brought together towards the smaller* (> <).

Thus, "$4 = 4$" means 4 *is equal to* 4; "$5 > 2$" means 5 *is greater than* 2; "$3 < 7$" means 3 *is less than* 7.

Place the proper character between

7 and $\frac{14}{2}$. $\frac{3}{25}$ and $\frac{4}{25}$.
0 and .0001.   1.001 and $1\frac{1}{1000}$.
$\frac{3}{4}$ and .75. $\frac{3}{5}$ and $\frac{3}{15}$.

3. *A character used instead of words, is called a* SIGN *or* SYMBOL.

Thus, =, <, >, are signs.

Point out and explain the signs in these examples:—

$3.75 = 3\frac{3}{4}$. $6.5 > 6$.
$7 < 15$. $3 = \frac{27}{9}$.

4. *When we explain a sign or symbol by words, we* INTERPRET *it.*

Thus, = means "equal to."
< means "less than."
> means "greater than."

5. *Two numbers united by a sign, form a combination called an* EXPRESSION *by symbols.*

Thus, "$10 = 10$" is an expression for the equality of two numbers; "$10 < 15$" or "$10 > 8$" is an expression for the inequality of two numbers.

NOTE.—All figures, as 1, 2, 3, 4, 5, &c., are symbols. Hence, any *expression* containing figures, or any other symbol, as "$17 > 15$" is *interpreted* when its meaning is given in words, as "seventeen is greater than fifteen." Hereafter, let the learner interpret every new expression.

6. *An expression for the equality of two or more numbers, is called an* EQUATION; *an expression for the inequality of two or more numbers, is called an* INEQUALITY.

Thus, "$12 = 12$" is an equation; "$3 < 8$" is an inequality.

What decimal may be put equal to each of these vulgar fractions?

$$\tfrac{2}{5}, \tfrac{3}{4}, \tfrac{7}{11}, \tfrac{8}{12}, \tfrac{13}{25}, \tfrac{17}{68}.$$

Put an improper fraction equal to each of the following mixed numbers:

$$3\tfrac{1}{5}, 7\tfrac{5}{7}, 8\tfrac{4}{11}, 12\tfrac{3}{8}, 13\tfrac{1}{17}.$$

SECTION II.

EQUATIONS CONTAINING THE SUM AND THE DIFFERENCE OF NUMBERS.

Jane has 12 pencils, and Julia has only 7. How many must be added to Julia's number to make it equal to Jane's?

Then, 5 added to $7 = 12$.

7. *We may form an equation by putting the* SUM *of two numbers equal to a third.*

Thus, 10 added to $17 = 27$.
13 added to $13 = 26$.

8. *Instead of the words "added to," we may use the sign* $+$, *called* PLUS, *to express the* SUM *of two numbers.*

Thus, "$5 + 3$" is an expression (5) for the *sum* of 5 and 3, and should be read 5 *plus* 3; and "$5 + 3 = 8$" is an equation containing the sum of two numbers.

Instead of the following blanks, put in such numbers as will form the equations.

12 + (23) = 35. (31) + 25 = 56.
(11) + 28 = 39. (77) + (26) = 103.
29 + 86 = (115). 1.64 + (3.46) = 5.
47 + (29) = 76. 75 + (67) = (142).

9. *When we join numbers by the sign +, we* INDICATE *their sum; when we* PERFORM *what is indicated, we obtain the sum, and thus* REDUCE *the expression to a single number.*

Thus, 8 + 10 + 16 + 20 is the *indicated* sum of 8, 10, 16, and 20. By performing the addition indicated, we obtain 54 as the *reduced* sum. Hence, 8 + 10 + 16 + 20 = 54.

Indicate the sum of 8, 7, 13, and 25 46 of 9, 11, 18, 27, and 34; of 1.04, 3.0006, 5.9637, and .34; of $\frac{3}{5}$, $\frac{8}{11}$, and $\frac{7}{10}$. Put each indicated sum equal to the reduced sum. =99 10.3443 54, 0 3 4 4 6 = $\frac{2}{13}$

Robert had 15 marbles, but lost 5. How many has he left?

Then, 15 less 5 = 10.

10. *We may form an equation by putting the* DIFFERENCE *of two numbers equal to a third.*

Thus, 17 less 7 = 10.
19 less 8 = 11.

11. *Instead of the word "less" we may use the sign —, called* MINUS, *to express the* DIFFERENCE *of two numbers.*

Thus, "6 — 3" is an expression (5) for the *difference* between 6 and 3; and "6 — 3 = 3" is an equation containing the difference of two numbers.

Instead of the blanks below, put in the numbers required to make the equations.

25 — (12) = 13. 36 — (35) = 1.
48 — (16) = 32. (25) — (13) = 12.
(28) — 11 = 17. 39 — 13 = (26).

12. *When we join numbers by the sign —, we* INDICATE *their difference; when we* PERFORM *what is indicated, we* *obtain the difference, and thus* REDUCE *the expression to a single number.*

Thus, 19 — 12 is the *indicated* difference between 19 and 12. By performing the subtraction indicated, we obtain 7 as the *reduced* or *actual* difference.

Indicate the difference between

7 and 15,	25 and 43,
19 and 25,	11.32 and 19.03,
8 and 31,	$13\frac{1}{3}$ and $15\frac{3}{12}$.

8 18 23

Make equations by putting each indicated difference equal to the reduced or actual difference.

Walter earned 12 cents at one time, and gave away 3; again, he earned 6, and gave away 2. How many had he left the first time? How many the second? How many in all? Indicate what he had left the first time; the second time; in all. Thus, he had 12 — 3 added to 6 — 2, or by (8) 12 — 3 + 6 — 2. How will you find how many he had left in all? *Ans.* By taking all he gave away; namely, 3 and 2, that is, 3 + 2, or 5, from all he earned; namely, 12 + 6, or 18. Hence, 18 — 5 = 13 is what he had left.

13. *To reduce to a single number an expression having some numbers joined by +, and some by —, take the* SUM *of the latter from the* SUM *of the former.*

Reduce to a single number

18 — 11 + 6 — 2 + 1,
9 + 8 + 7 — 6 — 21 + 4 — 2 — 2,
19 — 14 + 3 — 1 + 50 — 10,
13 — 6 + 3 — 5 + 1[illegible] — 1,
17 + 3 — 6 + 18 — 12 + 21 — 7,
19 — 18 + 1 — 2 + 12 — 8 — 4.

17+3+18+21=
6+12+7=
19+1+12=32

14. *The parts of an expression joined together by +, or —, are called its* TERMS; *those preceded by the sign + are called*

PLUS TERMS, *and those preceded by the sign* — *are called* MINUS TERMS. *A term without a sign is always plus.*

Point out the plus and the minus terms in the preceding exercise. Put each expression equal to the reduced result.

15. *We may form an equation by putting any number of plus and minus terms taken together, equal to a single term.*

Thus, $12 - 3 + 6 - 2 = 13$; and
$16 - 6 + 7 - 4 = 13$. But,

16. *Any two things which are equal to the same thing are equal to each other.*

Therefore $12 - 3 + 6 - 2$ must equal $16 - 6 + 7 - 4$, since each expression is equal to 13. Thus,

17. *We may form an equation having any number of plus or minus terms, on either side of the sign* $=$; *those on the left are called the* FIRST *or* LEFT HAND MEMBER, *and those on the right the* SECOND *or* RIGHT HAND MEMBER *of the equation.*

Determine which of the following expressions are equal to each other, and put them into equations:

$20 - 5 + 8 - 4 + 1$. = 20 $100 - 75 + 8 - 14$.
$3 + 2 + 10 - 4 + 2$. = 13 $25 + 2 - 8 + 3 - 2$.
$6 + 14 - 8$. $14 + 3 - 2 + 5 - 1$. 19
$18 - 1 - 8 + 4$. $4 + 15 - 2 + 8$.
$50 + 60 - 30 - 25 - 30$. $5 + 6 - 2 + 10 - 7$.

SECTION III.

KNOWN AND UNKNOWN QUANTITIES.

Frederic has walked 37 miles. How many miles more must he walk to travel in all 99 miles?

Here we have 37 + the part not given = 99.

18. *When a part is not given, but is required by such words as "How many," "Required," "What number," &c., it is called an* UNKNOWN QUANTITY. *All others are* KNOWN QUANTITIES.

Thus, in the following equations, the terms expressed by *what, how many*, and *what number*, are unknown.

28 + what = 40.
What number + 30 = 95.
37 + how many + 12 = 65.

19. *Instead of the words "what," "what number," &c., we may use some symbol* (3), *as x, y, or z, for the unknown quantity.*

Thus, "$x + 6 = 15$, means (5 note) "What number added to six is equal to fifteen," and should be read "x plus $6 = 15$."

Read and interpret the following examples:

$x + 12 = 16.$ $z + 25 = 96.$
$y - 13 = 20.$ $48 - z = 25.$
$x + 19 = 27.$ $17 + x = 30.$

We can reduce (9) 8 + 12 to the single term 20; or 16 — 5 to 11. To what single term can you reduce $x + 12$? or $y - 3$?

20. *The sum or the difference of any known number and an unknown quantity can never be reduced to a single term. It can only be indicated.*

Thus, $x + 12$ is the indicated sum (9) of 12 and the unknown quantity x; and $x - 12$ is the indicated difference.

Is the expression $x + 12$ known or unknown? Is any part of it known? Is it known as a whole?

21. *Any expression, as $x + 12$, or $x - 12$, containing an unknown quantity, is itself unknown.*

But, if $x + 12$ were equal to 56, we should have the equation

$$x + 12 = 56.$$

Now, separate 56 into two parts, making one of them 12. The equation will then be

$$x + 12 = 44 + 12.$$

And since the part 12 in the first member is equal to the part 12 in the second, what must the part x be equal to? Then,

$$x = 44.$$

Hence,

22. *To obtain the value of an unknown quantity, make it stand* ALONE *equal to a known quantity, called its* EQUIVALENT.

Find, as above, the value of x in the following equations:

$x + 37 = 99.$ $\qquad$ $88 + x = 90.$

$44 + x = 150.$ $\qquad$ $x + 21 = 77.$

$1 + x = 7.$ $\qquad$ $x + 36 = 48.$

In the equation $4 = 4$, if you add 6 to each member, will the results be equal or unequal? Will the value of each member be changed? Then, in an equation,

23. *We may change the* VALUE *of the members, and yet preserve their* EQUALITY.

Thus, take the equation $12 = 12$; first *add* 5 to both members; take the same and *subtract* 5; take the same and *multiply* by 6; *divide* the same by 6. In the same manner try any other equation with any other numbers. Are the results equal or unequal? Then,.

24. *If equals be* ADDED *to equals, the sums will be equal.*

25. *If equals be* SUBTRACTED *from equals, the remainders will be equal.*

26. *If equals be* MULTIPLIED *by equals, the products will be equal.*

27. *If equals be* DIVIDED *by equals, the quotients will be equal.*

These simple truths, called *axioms*, will aid in removing a known from an unknown quantity.

SECTION IV.

TRANSPOSITION.

In the equation $x + 35 = 72$, is the first member known or unknown? See (21). What must be taken from it to leave x alone? What must be taken from the second to preserve the equality? See (25). Then, in an equation,

28. *To remove a* PLUS *term from x,* SUBTRACT *its equal from both members.*

Thus, by subtracting 35 from $x + 35$, x is left alone; and by subtracting the same from 72, the equality is preserved. We may first indicate (12) the subtraction thus,

$$x + 35 - 35 = 72 - 35.$$

What two terms in the first member are exactly equal? What sign has each? Reduce them to a single term by (12). What remains? Then,

29. *A minus term cancels an equal plus term.*

Thus, when we wish to cancel a term, as 12, in the equation $x + 12 = 19$, we write after it -12, and to preserve the equality, write the same in the second member; or, since we know that $+12$ and -12 will cancel each other, we may omit them altogether in the first member, and write -12 in the second to preserve the equality. The new equation still has 12, not in the first member, but in the second, and with the sign *minus* instead of plus.

30. *When a term is thus changed from one side of the equation to the other, it is said to be* TRANSPOSED.

NOTE.—The learner should see that transposing a term is but the application of (24) and (25).

In the equation $x - 16 = 30$, is $x - 16$ known or unknown? How much must be added to $x - 16$ to cancel 16, and thus to leave x alone? See (29). How much must be added to 30 to preserve the equality? See (24). Then, in an equation,

31. *To remove a* MINUS *term from x,* ADD *its equal to both members.*

Thus, by adding 16 to $x - 16$, we get x alone; and by adding 16 to 30, we preserve the equality. See (24).

Indicating the addition of 16, the equation stands,

$$x - 16 + 16 = 30 + 16.$$

or by (29) $x = 30 + 16$, or 46.

Here, as before, the term -16 seems to be taken from the first member, and written in the second with the sign $+$, and thus is said to be transposed. Hence,

32. (1.) *To transpose a* PLUS *term, write it in the opposite member, with the sign* $-$; and

(2.) *To transpose a* MINUS *term, write it in the opposite member with the sign* $+$.

If x is found in the second member, as in the equation

$$16 = 40 - x,$$

we may transpose it to the first member, thus,

$$16 + x = 40.$$

This equation is the same as though we had added x to each member, thus,

$16 + x = 40 + x - x.$ Cancelling $+x$ and $-x$,
$16 + x = 40.$ Transposing 16,
$x = 40 - 16.$
$x = 24.$

Hence, we have the following general rule for transposition:

33. *Any term may be transposed from one member of the equation to the other by changing its sign.*

NOTE.—The unknown terms should always be transposed to the *first* member, and the known terms to the *second*.

Find the value of x in the following equations:

1. $x - 20 = 75$.
2. $x + 1 = 150$.
3 $x - 95 = 1$.
4. $x + 20 = 56$.
5. $20 = 80 - x$.
6. $x - 100 = 16$.
7. $x + 12 = 37 + 5$.
8. $x + 60 - 20 = 100$.
9. $80 - 30 = 70 - x$.
10. $40 = x + 15$.

In this last example, after transposing the unknown terms to the first member, and the known terms to the second, we obtain the equation

$$-x = -25.$$

If now we transpose x to the second, and 25 to the first member, we shall have

$$25 = x; \text{ or, what is the same thing,}$$
$$x = 25.$$

If we had changed the signs in the equation $-x = -25$, we should have had the same result. In the same way

34. *The signs of* ALL *the terms in any equation may be changed without destroying the equality.*

NOTE.—Whenever the first member is $-x$, it will be necessary to change *all* the signs in the equation.

Find the value of x in the following equations:

1. $90 = x - 60$.
2. $28 = x - 15$.
3. $x - 96 = 3$.
4. $40 = x - 73$.
5. $75 - 20 = x - 40 + 6$.
6. $100 - 75 = x - 6 + 20$.

SECTION V.

PROBLEMS.

If, in the equation $x + 16 = 24$, we should interpret (4) all its parts in common language (19), we should have the question:

"What number is there to which, if sixteen be added, the sum will be twenty-four?"

35. *When an example is thus given, not in symbols, but in common language, it is called a* PROBLEM.

Make problems from the following equations:

$x + 75 = 100.$ $x - 60 + 30 = 70.$

$x - 34 = 60.$ $x + 35 = 28 + 36.$

On the contrary, when an example is given in language, and we are required to obtain an answer, we must first express all its parts by symbols.

36. *When the parts or conditions of a problem are expressed by symbols, the problem* IS PUT INTO AN EQUATION.

Thus, take the problem,—

John and Charles gathered 25 quarts of nuts; of these John gathered 12 quarts. How many did Charles gather?

Here we have 12 + the number of quarts Charles gathered = 25. But this is partly in common language and partly in symbols. If, now (19), we let the symbol x represent what Charles gathered, we shall have

$$12 + x = 25.$$

Transposing 12, $x = 25 - 12.$

Reducing by (12) $x = 13.$

1. What number added to 56 will make 100?

Let x = the number.

Then, putting the question into an equation (36), we have

$$x + 56 = 100.$$

$$x = 44.$$

In the same way perform the following examples:

2. A pole, 36 feet long, stands 17 feet in water. How many feet are above water?

3. A man 40 years old, has passed 19 years in America,

and the rest of his life in England. How many years has he spent in England?

4. From what number, if 30 be taken, will 28 remain?

5. A merchant sold some flour for $1000, which was $80 less than he gave for it. How much did it cost him?

6. A laborer earned $100, of which he spent $65. How much had he left?

7. A gentleman being asked his son's age, replied, "If you add 28 years to my son's age, the sum will express my age, which is 40 years." Required his son's age.

8. What number is that, to which if 45 be added, the sum will be 81? 36

9. What number is that, from which if 18 be taken, 60 will remain?

10. A farmer sold his horse for $150, which was $25 more than it cost him. How much did it cost him?

SECTION VI.

FACTORS AND COEFFICIENTS.

A boy bought 5 melons for 50 cents. How much did he pay for each?

Let $x =$ what he paid for each (19).

Then, 5 times $x = 50$.

37. *To express the product of two numbers, place the sign* ($\times$), *or* (.), *between them instead of the word "times."*

Thus, instead of 5 *times* x, we may use $5 \times x$, $5.x$, or briefly $5x$. Each of these expressions means 5 times x, or 5 multiplied by x; and, in the case of the last, is read "five x." The equation given above is written

$$5x = 50.$$

Note.—When two *figures* are to be multiplied, the sign should never be omitted; thus, write 5×6, not 56. This last expression would be fifty-six, and not 5 times 6, or 30.

38. *The numbers or quantities, which unite to form a product, are called its* FACTORS.

Thus, 5 and x are the factors of $5x$; 6 and x are the factors of $6x$; 4, 5, and 7 are the factors of $4 \times 5 \times 7$, or 140.

39. *When the factors of a product consist of a figure and a letter, the figure is placed first, and is called a* COEFFICIENT.

Thus, in $12x$, $18x$, $9y$, $11z$, 12, 18, 9, and 11 are coefficients.

Point out the coefficients in

$7x$. $5y$. $8z$. $9w$.
$3x$. $7y$. $13z$. $17w$.

40. *When the coefficient of a letter is* 1, *it is not written.*

Thus, x is the same as $1x$; $y = 1y$; $z = 1z$.

41. *A product represented by its several factors is only indicated. It is obtained when the indicated operation is performed.*

Thus, $5 \times 10 \times 2 \times 7$ is the indicated product of 5, 10, 2, and 7; and 700 is the actual product, found by multiplying the factors together.

Indicate and then perform the multiplication

Of 10, 7, 9, and 4. Of 2, 3, 5, 7, and 11.
Of 16, 3, 11, and 2. Of 18, 16, 5, and 9.
Of 5, 11, 13, and 17. Of 7, 3, 13, 17, and 12.
Of 3, 3, 3, and 3.

42. *A product consisting of a single factor repeated, as* $3 \times 3 \times 3 \times 3$, *is called a* POWER *of that factor. The power is best indicated by placing above and to the right of the factor a small figure, called an* EXPONENT.

Thus, $5 \times 5 \times 5$ is the third power of 5; and 5^3, which is read "5 raised to the 3d power," or briefly "5 third power," is the same thing written with the exponent 3, to show how many times 5 is used as a factor.

Indicate and interpret (3) the 5th power of 7; the 8th power of 2; the 6th power of 4; the 10th power of 9; the 2d power of x.

In the expression $12x$, what is the coefficient? Is $12x$ known or unknown? Is any part of it known? Is it known as a whole?

If $12x$ were equal to 84, we should have the equation

$$12x = 84.$$

Now, separate $12x$ into two factors $12 \times x$, and also 84 into the factors 12×7; the factor 12 will be equal to the factor 12, therefore the factor x must be equal to the factor 7.

Hence, $$x = 7.$$

Or, if we divide $12x$ by 12, we shall obtain x. Then, to preserve the equality (27), we must divide 84 by the same. This gives us, as before,

$$x = 7.$$

Hence, we see that

43. *To cancel any factor of a product, we must divide the product by that factor.*

Thus, to cancel 8 in $8x$, we must divide by 8. What must you divide by to cancel the co-efficients in the following examples?

$36x$. $25x$. $11x$.
$28x$. $360x$. $175x$.
$160x$. $120x$. $19x$.

In an equation,

44. *To remove a* COEFFICIENT *from x, divide* (27) *both members by that coefficient.*

Thus, in the equation $16x = 64$, divide by 16, and we have

$$x = 4.$$

Remove the coefficients from x in these equations:

$13x = 39.$ $113x = 339.$

$7x = 49.$ $17x = 85.$

$8x = 72,$ $11x = 121.$

A boy bought a peach and a melon for 12 cents, and the melon cost three times as much as the peach. What was the cost of each?

Let $x =$ the cost of the peach (19),

Then $3x$ will be the cost of the melon, and

$$x + 3x = 12.$$

Here x is found in two terms, having different coefficients. (See 40.) To obtain one term having one coefficient, we must find how many times x, are once x and 3 times x. How many times 2 are once 2 and 3 times 2? How many times 5 are once 5 and 3 times 5? How many times x are once x and 3 times x? Then, if

$$4x = 12, \text{ by } (44)$$
$$x = 3.$$

How many times x are

$4x$ and $7x$? $6x$ less $2x$?

$6x + 3x$? $8x - 3x$?

$15x + 12x + 3x$? $13x - 4x$?

$10x + 5x + x$? $15x + 2x - 3x$?

$17x - 4x + 11x - 6x + 3x$?

45. *To reduce to a single term several terms containing x, unite in one number by* (13) *all the coefficients, and make the result the new coefficient of x.*

Thus, in the equation

$$x + 3x + 7x = 100 - 14x.$$

Transposing $14x$ by (33), $x + 3x + 7x + 14x = 100.$

Reducing (45), $25x = 100.$

Dividing by 25 (44), $x = 4.$

In the equation

$$17x + 3x = 63 + 12x + x.$$

Transposing, $17x + 3x - 12x - x = 63.$

Reducing, $7x = 63.$

Dividing, $x = 9.$

In the same way solve the following equations;

7x + 3

1. $7x + 10 = 5x + 24.$
2. $5x = 2x + 15.$
3. $8x + 8 = 40.$
4. $6x + 3 = 4x + 17.$
5. $10x = 42 - 4x.$
6. $4x - 7 = 33 - x.$
7. $6x + 8 = 40 - 2x.$
8. $7x - 10 = 9x - 36.$

4x

SECTION VII.

DIVISORS AND CLEARING OF FRACTIONS.

A gentleman, being asked his age, replied, that if his age were divided by 3, the quotient would be 14 years. How old was he?

Let x (19) represent his age.
Then x divided by $3 = 14.$

46. *To show that one number is* DIVIDED *by another, place the sign* ÷ *between them, instead of the words "divided by;" or, better, make the dividend the numerator, and the divisor the denominator of a fraction.*

Thus, $15 \div 3$, or $\frac{15}{3}$, means that 15 is divided by 3, and is read "15 divided by 3." In the second form, it may be read "fifteen thirds." In either case, it is the same as $\frac{1}{3}$ of 15. So x divided by 3 may be written indifferently, $x \div 3$, $\frac{x}{3}$, $\frac{1}{3}$ of x, or $\frac{1}{3}x$. So, $3x$ divided by 5 is the same as $3x \div 5$, $\frac{3x}{5}$, or $\frac{3}{5}x$.

47. *When the coefficient of a letter is a fraction, it means that the product of the letter by the numerator is to be divided by the denominator.*

Thus, $\frac{5}{7}x$ is the same as $5x \div 7$, or $\frac{5x}{7}$, and may be read "five-sevenths of x," "$5x$ divided by 7," or "$5x$ sevenths."

The equation given above may then be written,

$$\frac{x}{3} = 14, \text{ or } \frac{1}{3}x = 14.$$

In $\frac{x}{3} = 14$, is x alone? Is $\frac{x}{3}$ known or unknown? Is any part of it known? What do you call that part?

Now, if $\frac{x}{3}$, or $\frac{1}{3}$ of $x = 14$, the whole of x must be equal to 3 times $\frac{1}{3}$ of x, or 42. Hence, $x = 42$.

If we multiply $\frac{x}{3}$ by 3, we shall obtain $\frac{3x}{3}$, or $\frac{3}{3}$ of x, that is x. Then, to preserve the equality (26), we must multiply 14 by 3. This gives, as above, $x = 42$. Hence,

48. *To cancel the denominator of a fraction, multiply the fraction by that denominator.*

Thus, to cancel 10 in $\frac{x}{10}$, we multiply by 10.

What must we multiply the following fractions by, to cancel the denominators?

$\frac{2}{3}$. $\frac{x}{12}$. $\frac{x}{130}$.

$\frac{4}{5}$. $\frac{x}{15}$. $\frac{x}{163}$.

$\frac{6}{7}$. $\frac{x}{13}$. $\frac{x}{750}$.

Then, in an equation,

49. *To remove a denominator from x,* MULTIPLY (26) *both members by that denominator.*

Thus, in the equation $\frac{x}{5} = 2$, multiply both members by 5, and we have

$$x = 10.$$

Remove the denominators in the following equations:

$$\frac{x}{15} = 3. \qquad \frac{x}{150} = 1.$$

$$\frac{x}{10} = 5. \qquad \frac{x}{17} = 8.$$

$$\frac{x}{130} = 2. \qquad \frac{x}{13} = 9.$$

A gentleman, being asked how many sheep he had, replied, that $\frac{2}{7}$ of them added to $\frac{3}{5}$ of them was just 62. How many sheep had he?

Let $x =$ the whole number of sheep. Then

$$\frac{2x}{7} = \frac{2}{7} \text{ of the flock, and}$$

$$\frac{3x}{5} = \frac{3}{5} \text{ of it.}$$

$$\frac{2x}{7} + \frac{3x}{5} = 62.$$

Here x is found in two terms, with different divisors or denominators. In order to unite these terms, so as to apply (49), we may write the equation (47)

$$\frac{2}{7}x + \frac{3}{5}x = 62.$$

In this equation, the coefficients of x are $\frac{2}{7}$ and $\frac{3}{5}$. Taking the sum of these coefficients by (45), we have $\frac{2}{7}+\frac{3}{5}$, or, reducing to a common denominator, $\frac{10}{35}+\frac{21}{35}=\frac{31}{35}$. Hence,

$$\frac{31}{35}x=62. \qquad \text{Dividing by 31 (44),}$$

$$\frac{x}{35}=2. \qquad \text{Multiplying by 35 (49),}$$

$$x=70.$$

Again, if it is required to find the value of x, from the equation

$$\frac{x}{2}+\frac{x}{3}-\frac{x}{5}=19,$$

uniting the coefficients, we have $\frac{1}{2}+\frac{1}{3}-\frac{1}{5}=\frac{15+10-6}{30}$ $=\frac{19}{30}$. Hence,

$$\frac{19x}{30}=19. \qquad \text{Multiplying by 30 (49),}$$

$$19x=570. \qquad \text{Dividing by 19 (27),}$$

$$x=30.$$

The case is entirely similar, when x in some terms has integral coefficients, and in others fractional, as in

$$3x+\frac{1}{2}x+2x+\frac{1}{3}x=35.$$

Taking the sum of the coefficients by (45), we have $3+\frac{1}{2}+2+\frac{1}{3}$, or $5\frac{5}{6}=\frac{35}{6}$. Hence,

$$\frac{35x}{6}=35. \qquad \text{Multiplying by 6 (49),}$$

$$35x=210. \qquad \text{Dividing by 35 (44),}$$

$$x=6.$$

Instead of reducing the coefficients of x to a common denominator, and making the result the coefficient of x, it is usually more convenient to multiply the *whole* equation by this common denominator, which in the end will be the same thing. Thus, in the equation

$$\frac{2x}{7}+\frac{3x}{5}=62. \quad \text{Multiplying by 35 (26),}$$

$$10x+21x=2170. \quad \text{Reducing (45),}$$

$$31x=2170. \quad \text{Dividing by 31 (44),}$$

$$x=70, \text{ as before.}$$

Here $\frac{2x}{7}\times 35=\frac{70x}{7}=10x$; and $\frac{3x}{5}\times 35=\frac{105x}{5}$, or $21x$; or, as $\frac{1}{7}$ of 35 is 5, $\frac{2}{7}$ will be 2×5, or 10; and as $\frac{1}{5}$ of 35 is 7, $\frac{3}{5}$ will be 3×7, or 21.

In the equation

$$\frac{x}{2}+\frac{x}{3}-\frac{x}{5}=19. \quad \text{Multiplying by 30,}$$

$$15x+10x-6x=570. \quad \text{Reducing (45),}$$

$$19x=570. \quad \text{Dividing by 19,}$$

$$x=30.$$

By this method, we reduce the *whole equation*, as it were, to a common denominator, and then use the numerators to complete the operation. The case is the same where the *known* quantities contain fractions; thus, in the equation

$$x-\frac{2x}{7}=\frac{x}{2}+7\tfrac{5}{7}. \quad \text{Multiplying by 14,}$$

$$14x-4x=7x+108. \quad \text{Transposing (33),}$$

$$14x-4x-7x=108. \quad \text{Reducing,}$$

$$3x=108. \quad \text{Dividing by 3,}$$

$$x=36.$$

In the same way perform the following examples:

1. $x = \frac{x}{8} + \frac{x}{4} + 15.$ 4. $x = \frac{x}{2} + \frac{x}{3} + 36.$

2. $\frac{4x}{7} - \frac{2x}{5} = 24.$ 5. $\frac{x}{3} + 5 = \frac{x}{6} + 7.$

3. $x - \frac{2}{3} = 14 - \frac{x}{2} - \frac{x}{3}.$ 6. $x + \frac{1}{2} = \frac{3x}{4} + \frac{x}{6} + 3.$

In the 6th example, the common denominator is 48; but as the denominators 2, 4, and 6 have common factors, we may take the *least common multiple*, 12, for the *least* common denominator. If the denominators had been 2, 3, and 6, the least common multiple would have been 6.

Multiplying the equation $x + \frac{1}{2} = \frac{3x}{4} + \frac{x}{6} + 3$, by 12, we have

$$12x + 6 = 9x + 2x + 36. \quad \text{Transposing (33),}$$
$$12x - 9x - 2x = 36 - 6. \quad \text{Reducing,}$$
$$x = 30.$$

In this example, $\frac{3x}{4} \times 12 = \frac{3x \times 12}{4}$, or cancelling, $3x \times 3$; and $\frac{x}{6} \times 12 = \frac{12x}{6} = 2x.$

This work should be performed mentally; thus, ¼ of 12 is 3, and $3 \times 3x = 9x$.

In the equation $\frac{3x}{8} + \frac{5x}{6} - \frac{11x}{12} = 28$, the least common denominator is 24; multiplying by 24, we have

$$9x + 20x - 22x = 672. \quad \text{Reducing,}$$
$$7x = 672. \quad \text{Dividing by 7,}$$
$$x = 96. \quad \text{Hence,}$$

50. *To remove the denominators from an equation, multiply each term in both members by the least common multiple of the denominators.*

NOTE.—To remove the denominators, is the same as to *clear the equation of fractions.*

Find the value of x in the following equations.

1. $\frac{7x}{10}+\frac{4x}{5}-\frac{5x}{6}=40.$ y = 60.

2. $2x-\frac{5x}{9}-\frac{11x}{18}+\frac{x}{4}=78.$ x = 72.

3. $x-\frac{3x}{4}+\frac{3x}{14}=27\frac{6}{7}.$ [illegible]

4. $x+\frac{x}{2}-\frac{3x}{4}=\frac{5x}{6}-\frac{2x}{3}+6+\frac{5x}{12}.$ = 36.

5. $5x-\frac{5x}{2}=80-\frac{5x}{6}.$ x = 24.

6. $\frac{x}{14}+\frac{3x}{7}=42-\frac{x}{4}.$ [illegible]

7. $64-\frac{3x}{8}+\frac{5x}{6}=\frac{11x}{16}+\frac{5x}{12}+\frac{68}{3}.$ = 64,

8. $x-24\frac{1}{6}-\frac{x}{10}=\frac{40}{3}+\frac{2x}{5}.$ = 5.

9. $2x+\frac{2x}{9}-10-\frac{5x}{6}=3x-\frac{3x}{4}-\frac{17x}{18}-7.$ x = 36.

Make equations from the following problems:

10. A farmer sold equal quantities of corn and potatoes for \$16.00. Required the number of bushels of each, the corn being \$1.00, and the potatoes 60 cents per bushel. y = 10

11. Charles, Frank, and Edward had 60 marbles. Charles had 8 more than Frank, and Frank had 17 more than Edward. How many had each? y = [illegible], F 23. [illegible]

12. A grocer gave \$96 for 8 barrels of flour and 2 barrels of sugar, paying 4 times as much for the sugar as for the flour per barrel. Required the price of each per barrel.

13. The difference of two numbers is 3, and the greater added to 9 times the less is 83. What are the numbers?

14. In a certain school, $\frac{3}{5}$ of the whole number of scholars study arithmetic, and $\frac{2}{3}$ study spelling. There are 10 more scholars study spelling than arithmetic. Required the number of scholars in the school, and the number in each class.

15. What number is that, to which if $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ of itself be added, the sum will be 70?

16. Herbert spent $\frac{1}{3}$ of his money in Boston, $5 more than $\frac{1}{3}$ of it in Albany, and the remainder, which was $25, in New York. How much money had he?

17. A. has a certain sum of money; B. has $12 more than $\frac{4}{9}$ as much as A; C has $2 less than $\frac{5}{6}$ as much as A; and C has also the same amount as B. Required the sum each has.

+12 = 3

18. Multiply x by 7, divide the product by 9, add 2 to the quotient, and subtract 24 from the sum. The remainder will be $\frac{1}{6}$ of the value of x. What is the value of x?

SECTION VIII.

LETTERS.

James bought a hat and a cloak for $20. He paid for the cloak three times as much as for the hat. What did he pay for each?

Let x represent the price of the hat. Then

$3x =$ the price of the cloak, and

$x + 3x = 20$. Reducing (45),

$4x = 20$. Dividing (44),

$x = 5$, the price of the hat,

$3x = 15$, the price of the cloak, and

$5 + 15 = 20$. This shows the value of x to be correct.

Since $x = 5$, what would be the cost of both, if the cloak costs 4 times as much as the hat? 5 times as much? 6 times? 7 times? 8 times? 9 times? 10 times? 20 times? 50 times? 75 times? 100 times? 1000 times? *any number* of times as much? Thus, what will $x + 3x$ be equal to? $x + 4x$? $x + 5x$? $x + 6x$? $x + 7x$? $x + 8x$? $x + 9x$? $x + 10x$? $x + 20x$? $x + 50x$? $x + 75x$? $x + 100x$? $x + 1000x$? $x +$ *any number* $\times x$?

51. *To express an indefinite known number, we may use some symbol, as* a *or* b, *instead of the words "any number."*

Thus, "$x + ax$" means an unknown number added to *any number* of times, or a *given number* of times that unknown number; and $x + ax = b$ is an equation containing two such indefinite known numbers.

Again, if a or b stand for 1, 2, 3, 4, 5, and so on, we may at any time restore any one of these in place of each. Hence, the equation $x + ax = b$ may be written $x + 3x = 12$, $x + 10x = 33$, &c. We see, then, that a or b may be represented by any of the numbers 1, 2, 3, 4, 5, 6, 7, 8, and so on, indefinitely. Can 3 be used to represent any one of these numbers? Can it stand for 7? for 9? *Ans.* No. It can stand for *three*, and for no other number.

52. *To represent* DEFINITE KNOWN *numbers, we employ figures; but to represent* INDEFINITE KNOWN *numbers, we use the first letters of the alphabet, a, b, c, d, &c.*

Thus, if we were to add two definite numbers, as 6 and 8, we might first indicate their sum, $6 + 8$, and then unite them into one number, 14; but if we were to add two indefinite numbers, as a and b, we could, as before, indicate their sum $a + b$, but could not unite them. The indicated sum $a + b$ is, therefore, the only expression for the sum till each receives a definite value.

53. *An expression, composed of indefinite known numbers, so used as to indicate a result, is called a* FORMULA.

Thus, $a + b$ is a formula for the *sum* of any two numbers.

$a - b$ is a formula for the *difference* of two numbers.

ab is a formula for the *product* of two numbers.

$\frac{a}{b}$ is a formula for the *quotient* of two numbers.

If for a and b we put any one of the numerous values which they may represent, we may reduce each expression to a single number. Thus, if $a = 10$ and $b = 2$, $a + b = 12$, $a - b = 8$, $ab = 20$, $\frac{a}{b} = 5$.

54. *When definite numbers are thus put into a formula in place of letters, they are said to be* SUBSTITUTED, *and the formula can then be reduced.*

In the formula $a - b$, if we substitute 6 for a, and then successively 1, 2, 3, 4, &c., for b, we shall have

Formula.	*Reduced.*
$6 - 1$	$= 5.$
$6 - 2$	$= 4.$
$6 - 3$	$= 3.$
$6 - 4$	$= 2.$
$6 - 5$	$= 1.$
$6 - 6$	$= 0.$

Which gives the greater value, $6 - 1$ or $6 - 2$? $6 - 2$ or $6 - 3$? $6 - 3$ or $6 - 4$? $6 - 4$ or $6 - 5$? $6 - 5$ or $6 - 6$? Here we see, that in any expression, as $a - b$,

55. *The greater the number to be subtracted, the less the value of the expression.*

What is the value of the formula $a - b$, when we substitute as above, 6 for a and 6 for b? *Ans.* $6 - 6$, or by (29) 0, *zero.*

Now, if we substitute 7 for b, and 6 for a, will the value be greater or less than the preceding? See (55). Then, which is greater, $6 - 7$ or 0? How can $6 - 7$ be reduced? *Ans.* We can separate -7 into -6 and -1. Then the $6 - 6$ will cancel each other by (29), and — 1 will remain a number still to be subtracted, though there is no number from which to take it.

56. *All quantities preceded by the sign —, are called* NEGATIVE QUANTITIES; *all quantities not preceded by the sign —, are called* POSITIVE QUANTITIES.

Thus, — 3, — 11, — 25, are negative quantities; + 14, + 16, or merely 14, 16, are positive quantities. Which is the greater, 6 — 6 or 6 — 7? 6 — 7 or 6 — 8? 6 — 8 or 6 — 9? 6 — 9 or 6 — 10? See (55). Reduce each of these expressions. Then which is greater, 0 or — 1? — 1 or — 2? — 2 or — 3? — 3 or — 4? Then,

57. *In negative quantities, an* INCREASE *of number implies a* DECREASE *of value.*

Thus, $-2 > -5$, $-6 < -3$.

Write the proper sign between

— 7 and — 15.
— 1 and — 3.
0 and — 6.
2 and — 2.
— 1 and 1.
— 3 and 10.

Count from zero to 10. Do the numbers increase or decrease? Do their values increase or decrease? Are they positive or negative? Count from 10 down to 0. Do the numbers increase or decrease? Do their *values*? Then, are they positive or negative? Count from 0 to — 10. Do the numbers increase or decrease? Do their *values*? Are they positive or negative? Count from — 10 to 0. Which increase, the numbers or their values?

58. *In negative quantities, a* DECREASE *of number implies an* INCREASE *of value.*

Thus, in — 8, — 7, — 6, — 5, the numbers decrease, but the values they represent increase. Hence,

59. (1.) *A negative quantity is one which is to be taken in a sense directly opposite to that of a positive.*

(2.) *Two quantities, one of which is positive and the other negative, are unequal, even though expressed by the same numbers.*

Thus, $-7 < 7$ and $1 > -1$.

It will be seen, from the preceding sections, that questions which we have been accustomed to perform by Arithmetic may be performed by Algebra, but in a different way.

1. In Arithmetic, we perform all operations *directly* upon definite numbers; in Algebra, we first *indicate* the operation, whether upon definite or indefinite numbers.

2. In Arithmetic, we obtain as a result only the number which is found by *actually performing* all required operations; in Algebra, we obtain as a result a *formula* composed of *indicated* operations.

3. In Arithmetic, the number sought is never represented; in Algebra, it is always represented by some symbol, usually a letter, as x.

4. In Arithmetic, the answer or result appears by itself *alone*; in Algebra, it appears in an *equation*, as the *equivalent* of the symbol which was made to represent it.

5. In Arithmetic, the result does not show *what* previous operations have been performed; in Algebra, a *formula* discloses not only the *result* but all the *operations* by which it is to be obtained.

6. In Arithmetic, definite known numbers only are employed, and are represented by the figures 1, 2, 3, 4, &c.; in Algebra, any quantity may be represented by either definite or indefinite known numbers; for the latter the letters a, b, c, d, &c., are used.

7. In Arithmetic, positive numbers only are used; in Algebra, both positive and negative numbers are employed.

8. *Algebra,* then, is a branch of mathematics in which quantities are represented by letters, and operations are indicated by signs.

CHAPTER II.

OPERATIONS ON ALGEBRAIC QUANTITIES.

SECTION I.

REDUCTION OF POLYNOMIALS.

Definitions.

60. *Any quantity expressed by algebraic symbols, is called an algebraic quantity.*

Thus, a, $5a^2b$, $m^2 - n^2$, $a^3b^2 - x + 5mp$, are algebraic quantities.

61. *An algebraic quantity may be either a* MONOMIAL *or a* POLYNOMIAL.

(1.) A *monomial* consists of a single term; as, a, $5xy$, $m^3n^2x^5$.

(2.) A *polynomial* consists of two or more terms; as, $5a + 3xy$, $m^2p + n^2 + x + 3y$.

Polynomials are also called

(3.) *Binomials*, when they contain two terms only; as, $a + b$, $x^2 - y^2$.

(4.) *Trinomials*, when they contain three terms; as, $a^2b + 3c - 5m$.

62. *The terms of a polynomial may be taken together as one term, by including them in a* PARENTHESIS.

Thus, $a + b + c - d$ is a polynomial of four terms; but when included in a parenthesis, thus, $(a + b + c - d)$, we may regard it as a monomial, and perform operations on it accordingly.

Thus, $x + (a + b + c - d)$ is a binomial, of which x is the first term, and the quantity in the parenthesis, the second. So $n\ (a + b + c - d)$ expresses the product of two factors (38), n being one, and $(a + b + c - d)$ the other.

The *vinculum* is sometimes used instead of the parenthesis. Thus, $\overline{a + 3b} \times \overline{m^2n - y}$ represents the product of two factors; $a + 3b$ being one, and $m^2n - y$ the other.

63. *The terms of an algebraic quantity are either* SIMILAR *or* DISSIMILAR.

(1.) SIMILAR TERMS *are those which are exactly alike in their* LETTERS *and* EXPONENTS

Thus, a^2b^3, $5a^2b^3$, and $11a^2b^3$, are similar terms

REMARK.—Abstract numbers are always regarded as similar terms.

(2.) DISSIMILAR TERMS *are those which are unlike either in their letters or exponents.*

Thus, $5a^2n$ and $5a^2m^2$ are dissimilar terms.

It will be seen that two terms may be similar, and yet differ in their *signs*, their *coefficients*, and in the *order* of their letters.

Thus, $3a^5m^2$, $-7a^5m^2$, and m^2a^5 are similar terms.

So, also, two terms may be dissimilar, and yet agree in their *signs*, their *coefficients*, the *order* of the letters, and even in the *letters themselves*, provided the exponents of any one letter are unlike. Thus, $3a^3m^2$ and $3a^3m^3$ are alike in respect to their signs, their coefficients, the order of the letters, and the letters themselves; but, since they differ in the exponents of m, the terms are dissimilar.

It is usual, for the sake of uniformity, to arrange the letters in alphabetical order; thus, we say amx^2, not x^2ma.

Reduction of Polynomials.

64. *Algebraic quantities are said to be*

(1.) REDUCIBLE *when they can be changed from one form to another without change of value; or*

(2.) IRREDUCIBLE *when they cannot thus be changed.*

Thus, $2a + 3a$ may be reduced to $5a$; but $2a + 3b$ cannot be reduced.

65. *A polynomial is reduced to its simplest form, when it is changed to an equivalent expression, containing the least possible number of terms.*

Thus, $4 + 6 + 9 + 12$ can be reduced (13) to 31; and $x + 5x + 2x$ can be reduced (45) to $8x$; but the polynomial $a + b + c + d$ cannot be reduced. The polynomial $a + a + a + b$ reduces to $3a + b$; here we unite the three similar terms in one, and then annex the unlike term $+ b$. So, $a^2b + 3a^2b + m^2n + x = 4a^2b + m^2n + x$.

Let it be required to reduce the polynomial $5a + 6a - 3a - a + 2a$, in which all the terms are similar (63). Uniting the plus terms (13), we have $5a + 6a + 2a = 13a$, the sum of the plus terms; and $-3a - a = -4a$, the sum of the minus terms. But it has been shown (29) that a minus term will cancel an equal plus term; then, in the expression $+ 13a - 4a$, we may regard the $13a = 9a + 4a$, we shall then have $9a + 4a - 4a = 9a$, the expression reduced to its simplest form.

In the polynomial $10b - 8b + b - 6b - 3b$, the sum of the plus terms is $+ 11b$, and the sum of the minus terms is $- 17b$. Then $+ 11b$ must cancel $- 11b$, in the $- 17b$, therefore $+ 11b - 17b = - 6b$, the reduced form of the expression.

From these illustrations we derive the following rule for the reduction of polynomials:

66. (1.) *Unite the similar plus terms by adding their coefficients, and in the same manner unite the similar minus terms; then subtract the less coefficient from the greater,* *taking the result with the sign of the greater, as the coefficient of the common letter or letters.*

(2.) *To the reduced terms join all the dissimilar terms.*

Remark.—In long examples, it is convenient to arrange the similar terms under each other, crossing each term as we proceed, to prevent mistakes.

Thus, the polynomial $9x^2 - 4y^2 + 6z - mn^5 + 8 - 12b + 7 + 4b - 3x^2 + 7y^3 - 4mn^5 - x^2 - 8b + 7 - 5z + 2y^3 + 6x^2 + z + 9b - 18 + 7mn^5 - 10x^2 - 4 - 3b - 5y^3 - 2z - mn^5$, is reduced as follows:

$$\begin{array}{rrrrrr}
+9x^2 & -4y^3 & +6z & -mn^5 & +8 & -12b \\
-3x^2 & +7y^3 & -5z & -4mn^5 & +7 & +4b \\
-x^2 & & & & +7 & -8b \\
+6x^2 & +2y^3 & +z & +7mn^5 & -18 & +9b \\
-10x^2 & -5y^3 & -2z & -mn^5 & -4 & -3b \\
\hline
x^2 & 0 & 0 & mn^5 & 0 & -10b
\end{array}$$

Ans. $x^2 + mn^5 - 10b$.

We first arrange the given polynomial as directed above. It is well to count the terms after arranging, to be sure that none of the original terms of the polynomial are omitted.

Now uniting the similar terms in the first column by (66), we have $15x^2 - 14x^2 = x^2$.

In the same manner reduce the next column. The result is $-9y^3 + 9y^3$, which terms cancel (29), and we consequently omit the term containing y^3 in the result.

In the third column we have $7z - 7z = 0$. In the same way, $-6mn^5 + 7mn^5 = +mn^5$.

We then have $+22 - 22 = 0$; and $-23b + 13b = -10b$. Collecting these several results, the polynomial reduced becomes $x^2 + mn^5 - 10b$.

Reduce the following polynomials to their simplest forms:

1. $4b + 3b + 5b - 6b$.
2. $5az^2x + 3az^2x - az^2x - 7az^2x + 17az^2x$.

3. $-7xyz^2 + 12 + 3xyz^2 - 5 + 4xyz^2 - 13xyz^2$.

4. $4a^2x + 4y + 5a^2x - 3y - 3a^2x + 7a^2x - 6a^2x - 2a^2x + 9a^2x - y$.

5. $3ab + 2c^2 + 5ax - c^2 + 16 + 14 - 3ax + 5ab + 4c^2 - y$.

6. $16ab - 3x^2z + 9 - 5m^5 + 9x^2z - 3 + m^5 - 9ab + 4 - 3m^5 - 18ab - 8x^2z + 6m^5 + 12ab + 2x^2z - 11$.

7. $x^2y^3 + 14 - 13x^3y^2 + 12 + 3x^2y^3 - 8x^3y^2 + 15x^3y^2 - 20 - x^2y^3$.

8. $5a^2 - 5b^3 + x^2y - 3a^2 + 2b^3 - 5x^2y + 8xyz$.

9. $8x^3 - 7y^3 + 6m^4 + 4 - 2x^2 - 4y^3 - 9 - 2x^3 - 6z^5 + 9m^4 + 7m^4 - 9x^2 + 12 - 8y^3 + 4z^5 + 10x^3 - 4m^4 + 3y^3 - 6 + 4x^2$.

10. $19a^2b^2x^3 - 2amn^5 - 7bcd^2 - 3abc + 5bcd^2 + a^2mn^2 - 7abc + 6a^2bc - 10 + 3amn^5 - 6a^2bc - 6a^2b^2x^3 - 9bcd^2 - 6 - 13a^2b^2x^3 + 5 + 8bcd^2 + 9$.

SECTION II.

ADDITION.

67. ADDITION, *in Algebra, consists in finding a single polynomial which shall be the sum of several algebraic quantities taken together.*

These several algebraic quantities may be either monomials or polynomials. The result should always be reduced, when possible, to the simplest form (66). This reduction, however, forms no part of the addition; it affects only the *form*, not the value of the result.

Thus, the two monomials 6 and 4, are algebraically added by forming the polynomial $6 + 4$. This result reduced (63, Rem.), gives the number 10. If instead of numbers, it is required to add the letters a, b, and c, we should form the polynomial $a + b + c$. This is the algebraic or indicated sum (see 53); and, as the terms are dissimilar (64), it cannot be reduced to any simpler form.

Let the learner interpret each symbol (4), and then the expression for the sum. Also substitute (54) 12 for a, 16 for b, and 23 for c, and form first the algebraic or indicated sum, and then the arithmetical sum.

To add the four monomials a, a, a, a, we first form the polynomial $a + a + a + a$. This is the algebraic sum; but as the terms are similar (63), it may be reduced (66) to the monomial $4a$. In the same manner, $3a^2b$, $5a^2b$, and $10a^2b$, when added, give $18a^2b$ as their reduced sum. But a^2, $3a^2$, $5b^2$, and $3c^2$, when added and reduced, give the trinomial $4a^2 + 5b^2 + 3c^2$.

Suppose it is required to add the polynomial $b + c$ to the monomial a. We may indicate the sum thus, $a + (b + c)$. Here the polynomial is included in a parenthesis (62), and taken as a single term. Now as b and c are both to be added to a, when we remove the parenthesis, we join each term by its proper sign to a, and thus form the irreducible polynomial $a + b + c$. The same is true of all polynomials whose terms are plus. Thus, $3a^2$, $13a^2 + m^2$, and $2m^2 + a^2 + b$, give as their sum $3a^2 + (13a^2 + m^2) + (2m^2 + a^2 + b) = 3a^2 + 13a^2 + m^2 + 2m^2 + a^2 + b =$ (66) $17a^2 + 3m^2 + b$.

In the same way we may indicate the sum of a and $b - c$, as if each were a monomial; thus, $a + (b - c)$. Now if we remove the parenthesis, and add b to a, the sum $a + b$ is evidently too large; for it is not the whole of b, but b less c, that is to be added to a. Consequently we must take c from $a + b$, giving $a + b - c$ for the correct result. This is the same as if we had simply removed the parenthesis, and written each term within it, with its proper sign, after a. For instance, to add $15 - 6$ to 25, we may add 15 to 25, which gives a result 6 too great, and so we must take away 6; thus, $25 + 15$

$-6 = 34$. Or we may take 6 from 15, and then add 9 to 25, which also equals 34. The result is the same in both cases, but this last method is impossible when letters are used.

REMARK.—It is not essential to employ the parenthesis; but we may add the quantities, by simply writing them one after the other, giving each term its appropriate sign.

Let it be required to add $a^2b^2c - m^2p$, $5a^2b^2c - xy$, $3xy - 3a^2b^2c$, and $d - 2m^2p$. The sum will be $a^2b^2c - m^2p + 5a^2b^2c - xy + 3xy - 3a^2b^2c + d - 2m^2p = 3a^2b^2c - 3m^2p + 2xy + d$.

To add the two monomials $4a$ and $-a$, we must first return to the origin of $-a$. It results (55) from the binomial $a - 2a$, to which it is equal. By the previous example, we see that the sum is $4a + a - 2a$. This reduced, becomes $3a$, The same result would be obtained by writing the two monomials together with their proper signs. Thus, $4a - a = 3a$.

From this example it appears that addition, in algebra, does not, as in arithmetic, always denote *increase*. The algebraic *sum* of two numbers often means the same as the arithmetical *difference*. Thus, the sum of 10 and -6, is $10 - 6 = 4$. The number 4 is the algebraic sum; but in arithmetic it would be called the difference of the two numbers. Hence, in all cases, to add algebraic quantities,

68. *Write the quantities to be added one after the other, giving to each its proper sign, and then reduce the result by uniting similar terms.*

1. Add xy, $7xy$, and $-10xy$.
2. Add $73b^2cx$, $-60b^2cx$, $+2b^2cx$, and $-15b^2cx$.
3. Add $-mxy^3$, $16mxy^3$, $-20mxy^3$, and $2mxy^3$.
4. Add $14ab^5x$, $-6ab^5x$, $-7ab^5x$, $-3ab^5x$, $7ab^5x$, $-5ab^5x$, $3ab^5x$, $-2ab^5x$, $-7ab^5x$, $3ab^5x$, $7ab^5x$, $-ab^5x$, and $-8ab^5x$.
5. What is the sum of $6abx$, $-5a$, $-xy$, and 27? $6abx - 5a -$

6. Add $19a^2x$, $-3amn + xyz$, $-am$, and $-5a^2x$.

7. What is the sum of $15axy$, a^2b^2, $-2bmn$, $-12axy$, -24, $3bmn$, 17, $-a^2b^2$, and $3axy + yx^3$?

8. What is the sum of $5ab^3c$, $12x^7$, $-14ax$, $-4ab^3c$, $-16x^7$, $3y$, $-5ax + 76$, $-abc$, $5x$, 83, and $16ax$?

9. What is the sum of $-3hm$, 54, $9hm$, and -60?

10. Add $14ax^2$, $3b^5$, $2x$, $17y$, 37, $-4b^5$, $-15ax^2$, $-12y$, $+ax^2$, 37, $-5y^5$, -74, and $-x$.

11. Add $8xyz - x^2y^2$, $-3xyz$, $-9y^3z$, $754 + 7y^3z$, $-64x^2y^2$, and $-y^3z$.

12. Add $5ax + 3bc$, $7ax + 17y$, and $-3ax - 4bc$.

13. Add $8abc - 5a - 5 + xyz$, $3a - 4abc + 4xyz - 10$, $2abc + 9 - 15xyz + 24$, and $7xyz - 2abc + 4a - 18$.

14. Add $5a^8 - 17 - 7a^8 + 3x^2y$, $4x - 13 - 10x^2y + 21$, $8a^8 + 8x - 2x^2y + ax$, and $8x^2y + 9 - 3a^8 - 10x$.

15. Add $9axy - 15 + a^2 - ab - 9bc$, $bc - 8a^2 - 8ab + 36 - 5axy$, $4a^2 + 4ab - 5bc + axy - 13$, and $2 - 4axy + 8bc + 5a^2 + 5ab$.

16. Add $3a^3 + bc^2$, $5a^3 + 2bc^2$, $a^3 + 5bc^2$, and $6a^3 + 2bc^2$.

17. Add $a^2b^3 + 7 + x^2$, $3a^2b^3 + 9$, $x^2 + 14$, $3a^2b^3 - 19 - 5x^2$, $12 - 2a^2b^3$, and $4a^2b^3 - 8 + 2x^3$.

18. Add $a^3b^2 + 36a^6 + a^3y^4$, $3a^3b^2 + 12a^6 + 3a^3y^4$, $2a^3b^2 - 24a^6 + 2a^3y^4$, and $8a^3b^2 - 24a^6 + 8a^3y^4$.

SECTION III.

SUBTRACTION.

69. SUBTRACTION, *in Algebra, consists in finding a single polynomial which shall be the difference of two algebraic quantities.*

These quantities, as in addition, may be either monomials or polynomials. The resulting polynomial should always be reduced to its simplest form.

Thus, to take 7 from 13, we should have the polynomial $13 - 7 =$ (63, Rem.) 6. Here 13 is written with its proper sign (+ understood), and 7, that is $+ 7$, is written after it, with its sign changed from + to —. In the same way, to take b from a, we form the polynomial $a - b$, which cannot be reduced. Here the sign of b, the term to be subtracted, is changed from + to —. So to subtract $2a^2b$ from $3a^2b$, we have $3a^2b - 2a^2b =$ by (66) a^2b.

Let it be required to subtract $7 + 5$ from 20. It is obvious that $7 + 5 = 12$, and 12 from 20 leaves 8. Now let us place $7 + 5$ in a parenthesis, and indicate that it is to be taken from 20; thus, $20 - (7 + 5)$. If we remove the parenthesis, and subtract 7 from 20, the remainder $20 - 7$ or 13 is evidently too large, since it is not 7 alone, but $7 + 5$, that is to be taken from 20. If now we subtract 5 more, we shall have $20 - 7 - 5 = 8$ for the true remainder.

To subtract $b + c$ from a, we shall have $a - (b + c)$. Removing the parenthesis, and subtracting b, the remainder $a - b$ is too large, since $b + c$ is to be taken from a. Subtracting c, as above, we have $a - b - c$ for the true remainder. It will be seen in the preceding examples, that when we remove the parenthesis, the sign of each term of the quantity to be subtracted (or subtrahend) is changed from + to —, and written after the minuend.

Suppose we have to subtract $b - c$ from a. Using the parenthesis, as above, we have $a - (b - c)$. Now if we remove the parenthesis, and subtract b from a, the result $a - b$ is too small by c, since we have taken away a quantity too large by c, for it is not the whole of b, but b less c, that we have to take away. Consequently, we must add c to correct the error. The result $a - b + c$ is the same as if, without the parenthesis, we had at once changed the sign of each term of the subtrahend $b - c$ from $+$ to $-$, and from $-$ to $+$, and written it after the minuend a, with the signs thus changed.

Let $a = 17$, $b = 12$, and $c = 8$; then $a - (b - c) = 17 - (12 - 8)$. Now, in arithmetic, we should say 12 less 8 is 4, and 4 from 17 leaves 13. But if we perform the operation indicated above, by the method employed in algebra, we shall first subtract 12 from 17; the result $17 - 12$ or 5 is evidently too small, for it is 12 less 8 that is to be subtracted from 17. Hence, we must add 8, and the result $17 - 12 + 8$, reduced, gives 13, as above.

It will be seen that it is not *essential* to employ the parenthesis, though it is a very convenient way of indicating the operations to be performed. The sign $-$, which precedes the parenthesis, does not belong to any particular term within it, but shows that the *whole expression* is to be subtracted.

In the above example, $a - (b - c)$, b has the sign $+$ within the parenthesis, which is changed to $-$ when the subtraction is performed. When we subtract at once, without first indicating the operation, we must change the signs of all the terms to be subtracted. Thus, if we have to subtract $3m^2 - 4x^2 + 6ay^3$ from $x^2 - 5m^2$, we shall have $x^2 - 5m^2 - 3m^2 + 4x^2 - 6ay^3 = 5x^2 - 8m^2 - 6ay^3$.

To subtract $- a$ from $4a$, we may take the equivalent of $- a$ (55 ex.), which is $a - 2a$. This taken from $4a$, gives $4a - a + 2a = 5a$, a quantity greater than $4a$ by a. We should have obtained the same result by changing the sign of $- a$, and writing it after $4a$, thus, $4a + a = 5a$.

It will be seen from this example, that subtraction, in algebra, does not always, as in arithmetic, imply diminution. The algebraic difference of two numbers is often the same as their arithmetical sum. Thus, to subtract $- 4$ from 5, is the same as to add $+ 4$ to 5; that is, it equals 9. That this is the true difference, may be seen by counting from $- 4$ to 0,

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which gives 4 units, while from 0 to 5 gives 5 more, or 9 units in all.

From the foregoing examples and illustrations, we have the following rule for subtraction:

70. *Change the signs of all the terms in the quantity to be subtracted; and write it, with the signs thus changed, after the quantity from which it is to be taken, reducing as in addition.*

1. Subtract $x - m$ from $5m$.
2. Subtract p from $7x^2$.
3. From xyz take xy.
4. From $8xy$ subtract $-y$.
5. Subtract $-2m^5$ from $8m^5$. + 7
6. From $an + ap$ subtract $xyz - 60$. an + ap − xy
7. From $x^2 + y^2$ take $x^2 - y^2$.
8. From $24x + 3yz - 12abc - 6m^3np + 5$ subtract $16abc - 6m^3np + 23x + yz + 12 - xy$.
9. Subtract $x - y - z$ from $x + y + z$.
10. Subtract $3abx - 14 - 6a^8$ from $3abx - 14 + 6a^8$. 12
11. Subtract $8a^2m - z$ from $13a^2m - zy$. z
12. Subtract $-3a^2$ from $3a^2b - 4x^2y^2 + 5a^2 - 7xyz$. 8

Perform the operations indicated in the following examples:

13. $48 - (36 + 2y)$.
14. $ab + (y - x)$.
15. $17xyz + 14 - a^2b - (13a^2b + 24 + 18xyz)$.
16. $-102 + 17xy - 12abm + (47abm - xy + 3am)$.
17. $abx + 3yz - (3yz + abx)$.
18. $5am - 17 - (-18 + 2am)$. 20 am

Remark.—It is often convenient to reverse the preceding operations in addition and subtraction, and return from an expression given as the *sum*, or as the *difference* of several quantities, to the quantities themselves from which it may have been derived.

Thus, $a + b + c$ may have been produced by adding a, b, and c together, or by adding $(a + b)$ to c, or a to $(b + c)$, or b to $(a + c)$.

As, when any quantity is subtracted, all its signs are changed (70), we must be careful, in restoring a quantity already subtracted, to change back its signs to those it had at first. Thus, $a - n + p$ may have resulted from subtracting $n - p$ from a, for $a - (n - p) = a - n + p$. So $m + 3x - y = m - (-3x + y)$.

Hence, when we enclose terms that have been subtracted in a parenthesis, we must *change the sign of each term,* and place the minus sign before the parenthesis.

We cannot discover from the result how many or what similar terms were reduced. $3am$, for instance, may have resulted from $am + 2am$, or from $7am - 4am$, or from $10am - 8am + am$, &c. Hence, we may put any number of terms in place of a single term, provided that, when reduced, they will be equal to that term.

Again, we cannot determine what terms had equal coefficients, and hence wholly disappeared in the result. Thus, $5am$ may have resulted from $8am - 3am + 5x^2 - 3x^2 - 2x^2$.

Hence, we may, in any result, introduce terms at pleasure that cancel each other, and not affect the value of the given expression.

19. Derive $a^2 - b + c$ from a^2 and $b - c$.

20. What two polynomials may be added to produce $3ax$ -4c $- 4c^2$?

21. Let $5m^2n - 3p$ be the difference between two polynomials, and let $3x^2y$ be found in both.

22. In the polynomial $3a^2 - m^2 + 3y - 1$, make the last three terms the subtrahend, and $3a^2$ the minuend.

23. Separate $25ay - 8ax^2 + 7yz + mn$, so that the last two terms shall appear as a subtrahend.

24. Make any expression, consisting of three polynomials, which, when added and reduced, shall equal a^2.

SECTION IV.

MULTIPLICATION OF MONOMIALS.

71. MULTIPLICATION, *in Algebra, consists in finding a quantity which shall be equal to any number of times a given quantity.*

The quantity thus formed, is called the *product.*

The quantity which is taken a given number of times, is called the *multiplicand.*

The quantity which shows how many times the multiplicand is taken, is called the *multiplier.*

The multiplier and multiplicand are both called *factors* (38) of the product.

Thus, in this example, $4 \times 3 = 12$, we have 4 for the multiplicand, 3 for the multiplier, and 12 for the product. 4 and 3 are factors of 12, and may be written thus, $3 \times 4 = 12$. Hence, it is immaterial which factor is taken as the multiplier, and which as the multiplicand; the product being the same in either case. 4×3 is the indicated product, and 12 is the product with the multiplication performed.

In the same way, we have 5×6, 5×7, $5 \times a$, or $5a$; $a \times 5$, $a \times 6$, $a \times b$, or ab. So the three factors a, b, and c give $a \times b \times c$, or abc, for their product. The product of a, a, a, and a is $a \times a \times a \times a = a^4$ (42). Here a is used four times as a factor, and the product is called the *fourth power* of a. So $b \times b \times b = b^3$, that is the third power of b. d^5 means that d is used five times as a factor.

72. *All algebraic quantities are either* PRIME *or* COMPOSITE.

(1.) *A prime quantity has no other entire factor than itself and unity.*

Thus, a, 7, 13, x, and $(a + m)$, are all prime.

(2.) *A composite quantity contains other factors than itself and unity.*

Thus, $15 = 5 \times 3$; $7a = 7 \times a$; $ab = a \times b$; $m^2 = m \times m$; $ax + bx = (a + b)\,x$, &c.

Remark.—The learner should carefully distinguish between a composite quantity and a polynomial. Thus, abc is a composite quantity, since it equals $a \times b \times c$; yet it is a monomial, since it consists of a single term. But $a + b + c$ is a prime quantity, since it has no divisor but itself and unity; and is a polynomial, since it has more than one term. In the polynomial $2a + 3b^2 + 6c$, each term is composite, yet the polynomial as a whole is prime, since there is no factor that will divide all the terms.

To multiply $3a^5b$ by $2m^2n^3x$, let us write out the factors which compose the terms; thus, $3 \times a^5 \times b \times 2 \times m^2 \times n^3 \times x = 6a^5bm^2n^3x$. We should have obtained the same result by taking the product of the coefficients, and then annexing the literal factors with their respective exponents, arranging them alphabetically.

To multiply $5a^3mx$ by $7a^2bm$, we shall have, as above, $5 \times a \times a \times a \times m \times x \times 7 \times a \times a \times b \times m = 35a^5bm^2x$. That is, when the same letter occurs in both factors, we give it an exponent in the product equal to the sum of its original exponents. Thus, $n^2 \times n^4 = nn \times nnnn = n^6$; $b \times b^3 = b^4$; $x^3 \times x^5 = x^8$, &c.

Remark.—Care should be taken to mark the difference between an exponent and a coefficient. Thus, $3a$ means $3 \times a$, and is equivalent to $a + a + a$; whereas a^3 means a used three times as a factor, and is equivalent to $a \times a \times a$. If we substitute 5 for a, $3a = 3 \times 5 = 15$; while $a^3 = 5^3 = 5 \times 5 \times 5 = 125$.

Hence, we have the following rule for the multiplication of monomials affected by the sign +:

73. *To the product of the coefficients, annex, in order, all the different letters found in the several factors, giving to each letter an exponent, equal to the sum of all its exponents, in all the given factors.*

What is the product of $5x^2y$, $7a^3mx^3$, and $2ay$?

Ans. $70a^4mx^5y^2$.

1. Multiply $3a^2b^2y$ by $6ab^3x$.
2. Multiply $8a^3b^2x$ by $2a^2b^2x$.
3. Multiply $3a^5xy$ by $5ab^3x^2y^5$.
4. Multiply $9a^3c^2m$ by $9a^3c^2m$. $81a^6c^4m^2$
5. Multiply amn by xyz.
6. What is the product of abc, hmn, and axy?
7. What is the product of y, $9mn^2$, and $3yz$?
8. What is the product of 14, $8x^2y^2z$, and $2ax$?
9. Multiply $8a^3b^3c^3$ by 3.
10. Multiply $2ax^2$ by $3a^2x$.

SECTION V.

SIGNS IN MULTIPLICATION.

To determine what sign shall be given to the product in all cases, let it be observed that,

74. *With a given* POSITIVE *multiplicand, the greater the multiplier, the greater the product; and the less the multiplier, the less the product.*

Thus, $5 \times 4 = 20$; $5 \times 5 = 25$; $5 \times 3 = 15$, &c.

Taking, then, any number, as 4, for a multiplicand, and any other number, as 3, for a multiplier, we have

$$4 \times 3 = 12.$$
$$4 \times 2 = 8.$$
$$4 \times 1 = 4.$$
$$4 \times 0 = 0.$$

It will be observed, that as the multiplier decreases successively by a unit, the product decreases by 4, that is, by as many units as are in the multiplicand; and also, that the products from 12 down to 0 are positive. Hence,

75. *A positive quantity, multiplied by a positive quantity, gives a positive product.*

Now continuing to diminish the multiplier as above, we shall have

$$\begin{aligned} 4 \times 0 &= 0. \\ 4 \times -1 &= -4. \\ 4 \times -2 &= -8. \\ 4 \times -3 &= -12. \end{aligned}$$

Here the same law still prevails; that is, as the multiplier decreases by a unit, each product is 4 less than the preceding. If 4 is multiplied by 1 less than 0, or -1, the product must be 4 less than zero, or -4; and so on in the successive products. Hence,

76. *A positive quantity, multiplied by a negative, gives a negative product.*

Since the *value* of negative numbers decreases, as the numbers themselves increase (57), let it be observed that,

77. *With a given* NEGATIVE *multiplicand, the greater the multiplier, the less the product; and the less the multiplier, the greater the product.*

Thus, $-5 \times 4 = -20$; $-5 \times 3 = -15$, a number less than -20, and hence representing a greater value (57); $-5 \times 5 = -25$, a number greater than -20, and consequently less in value.

Now taking any negative quantity, as -4, for a multiplicand, and taking 3 as a multiplier, we shall have

$$\begin{aligned} -4 \times 3 &= -12. \\ -4 \times 2 &= -8. \\ -4 \times 1 &= -4. \\ -4 \times 0 &= 0. \end{aligned}$$

Here, as the multiplier decreases successively by a unit, we find the product increasing by 4, the number of units in the multiplicand. Indeed, it is obvious that 3 times — 4 must give a less value than 2 times — 4. It will be observed that all these successive products are negative. Hence,

78. *A negative quantity, multiplied by a positive, gives a negative product.*

Now continuing to diminish the multiplier, as above, we shall have, by the principle in (77),

$$-4 \times 0 = 0.$$
$$-4 \times -1 = 4.$$
$$-4 \times -2 = 8.$$
$$-4 \times -3 = 12.$$

Here, as we multiply by — 1, which is a multiplier less than 0 by a unit, the product, as above, ought to be 4 greater than the preceding; that is, 4 greater than 0. So, if we multiply by — 2, a multiplier less than — 1 by a unit, the product must be 4 greater than the preceding; and so of all the rest. It will be seen that all the products, from 0 to 12, are positive. Hence,

79. *A negative quantity, multiplied by a negative, gives a positive product.*

It will be seen, (76) and (78), that the products are negative when the signs of the factors are *unlike;* and, (75) and (79), that the products are positive when the signs of the factors are *alike.* Hence, we have the following simple rule for the signs in multiplication:

80. *When the two factors have like signs, that is, both + or both —, the product has the sign +; but when they have different signs, that is, one + and the other —, the product has the sign —.*

1. Multiply $5ab$ by $3a$. $= + 15a^2b$

2. Multiply $-6xy$ by $3x$.
3. Multiply $7m^2n$ by $-5amn$.
4. Multiply $-a$ by $-b$; $+a$ by $-b$.
5. Multiply $-a$ by $+b$; $+a$ by $+b$.
6. Multiply mxz by -5.
7. Multiply -7 by a^2bc^3.
8. Multiply $-3a^2$ by $-5ab$.
9. What is the product of 7, mn^2, and a?
10. What is the product of $5py$, $7x^2y$, and $2ap$?
11. What is the product of $3a^2b^2cd^3$, 5, and $a^2b^2cd^3x$?
12. Multiply together a, $-2b$, $3c$, and d^3.
13. Multiply together -5, m, m^2, 3, $-m^3$, and 4.
14. Multiply $-2amn$ by $7xy$.
15. Multiply together x, $-xy$, xyz, and $-ax$.

SECTION VI.

MULTIPLICATION OF POLYNOMIALS.

To indicate the multiplication of a polynomial by a monomial, we may include the polynomial in a parenthesis (62), and regard it as a single term. Thus, $(a+b)\times c$, or $(a+b)c$, indicates the product of $a+b$ by c. To perform the multiplication here indicated, we must multiply a by c and b by c, and then add the results. Hence, $(a+b)c = ac + bc$.

If $6+4$, or 10 be multiplied by 2, the product will be 20; or, as above, $2(6+4) = 2\times 6 + 2\times 4 = 12 + 8 = 20$.

To indicate the product of $a-b$ by c, we have $(a-b)c$. Here we must take c times a, and then *subtract* c times b, since it is not a, but a less b that we multiply by c. Hence, $(a-b)c = ac - bc$.

If $12-7$, or 5, be multiplied by 3, the product will be 15; or, $(12-7)3 = 12\times 3 - 7\times 3 = 36 - 21 = 15$, as before.

To indicate the product of one polynomial by another, as $a + b$ by $c + d$, we include each factor in a parenthesis; thus, $(a + b)(c + d)$. To perform the multiplication here indicated, we first multiply $a + b$ by c, and then by d. The separate results, called *partial products*, are then to be *added* together, since we multiply by c *plus* d. Then $(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd$.

If we multiply *by* $c - d$, we must *subtract* $(a + b)d$, since we take $a + b$ not c times, but c *less* d times. Thus, $(a + b)(c - d) = (a + b)c - (a + b)d = ac + bc - ad - bd$, changing the signs of the quantity to be subtracted by (70).

$5 + 3 = 8$ multiplied by $6 + 4 = 10$ equals 80; or, as above, we shall have $(5 + 3)6 + (5 + 3)4 = (30 + 18) + (20 + 12) = 48 + 32 = 80$.

So $(5 + 3)(6 - 4) = 8 \times 2 = 16$, or $(5 + 3)6 - (5 + 3)4 = (30 + 18) - (20 + 12) = 30 + 18 - 20 - 12 = 16$.

$a - b$ multiplied by $c - d$, or $(a - b)(c - d) = (a - b)c - (a - b)d = ac - bc - ad + bd$. Here we multiply $a - b$ by c, and then subtract d times $(a - b)$.

So $(10 - 7)(5 - 3) = 3 \times 2$, or 6; or, as above, $(10 - 7)(5 - 3) = (10 - 7)5 - (10 - 7)3 = 50 - 35 - 30 + 21 = 71 - 65 = 6$.

Hence, it will be seen that, in the multiplication of polynomials, we simply multiply each term of one factor by each term of the other; that is, we really multiply monomials by monomials, following the rules in (73) and (80) for the exponents and signs, and then unite these partial products.

As the multiplication will often give rise to similar terms, which must be reduced, it is convenient to place the factors under each other in multiplying, and especially to place the *similar terms* in the partial products under each other, as in the following examples, where $4a^2 - 16ax + 3x^2$ is multiplied by $5a^3 - 2a^2x$:

$$\begin{array}{rrrrl}
4a^2 & -16ax & +3x^2 & & \\
 & 5a^3 & -2a^2x & & \\
\hline
20a^5 & -80a^4x & +15a^3x^2 & & \text{Product by } 5a^3. \\
 & -8a^4x & +32a^3x^2 & -6a^2x^3 & \text{Product by } -2a^2x. \\
\hline
20a^5 & -88a^4x & +47a^3x^2 & -6a^2x^3 & \text{Sum of partial products.}
\end{array}$$

We have, then, the following rule for the multiplication of polynomials:

81. *Multiply each term of the multiplicand by each term of the multiplier, observing that like signs give +, and unlike signs give — in the product.*

Add together the partial products, and reduce the similar terms.

1. Multiply $x + xy + y$ by $x - xy + y$.

$$\begin{array}{llllll}
x + xy + y & & & & & \\
x - xy + y & & & & & \\
\hline
x^2 + x^2y + xy & & & & & \\
\quad - x^2y & & - x^2y^2 - xy^2 & & & \\
\qquad\quad + xy & & + xy^2 + y^2 & & & \\
\hline
x^2 + 2xy - x^2y^2 + y^2. & \textit{Ans.} & & & &
\end{array}$$

2. Multiply $a^3 + a^2y + ay^2 + y^3$ by $a - y$.

$$\begin{array}{rrrrr}
a^3 & + a^2y & + ay^2 & + y^3 & \\
 & & a & - y & \\
\hline
a^4 & + a^3y & + a^2y^2 & + ay^3 & \\
 & - a^3y & - a^2y^2 & - ay^3 & - y^4 \\
\hline
a^4 & & & & - y^4. \quad \textit{Ans.}
\end{array}$$

3. Multiply $cd + b + c + a + x$ by ab.
4. Multiply $5axz + 1 + 3x - y$ by ax^2.
5. Multiply $b + c + 8$ by $b - c + 10$.
6. Multiply $4bx + 2b^2 + 5x$ by $3x - 7b$.

7. Multiply $a^2x - 4 - x$ by $-ax$.
8. Multiply $a^2 - bx^3 - xy^4$ by cmn.
9. Multiply $x + y + z$ by $x - y - z$.
10. Multiply $abx + cm$ by $1 - 5y^3$.
11. Multiply $a + 5 + x$ by $a - x - 5$.
12. Multiply $a^3 - b^3$ by $a^3 + b^3$.
13. Multiply $x^4 + x^3y + x^2y^2 + xy^3 + y^4$ by $x - y$.
14. Multiply $a^3 + a - 6$ by $2a^2 + a + 1$.
15. Multiply $a^2 + a^4 + a^6$ by $a^2 - 1$.
16. Multiply $a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4$ by $a + 2b$.
17. Multiply $2 + 6a^2 - b^3$ by $x^2 - a$.
18. Multiply $2a^2 - 3ax + 4x^2$ by $5a - 6a^2x^2 - 2x$.
19. Multiply $16a^2m + 3a^2$ by $-ay + 4m$.
20. What is the product of $x^2 + y^3 - 7$ by $x - y$.
21. What is the product of $3a + 5 - 2b^2$ by $5a + 2b^2 - 5$.
22. What is the product of $a^2 - 2ay + y^2$ by $a - y$.

SECTION VII.

COMBINATIONS OF FACTORS.

In algebraic operations, it is frequently very convenient to use the indicated product; that is, the product in which the factors are not involved with each other, but are kept separate and distinct. In all such cases, each polynomial factor must be included in a parenthesis. Thus, $a + b + c$ multiplied by d would be $(a + b + c)d$. If the parenthesis were not used, we should have $a + b + c \times d = a + b + cd$, instead of, as above, $ad + bd + cd$. $(m + n)(p + g) = mp + np + mg + ng$; while $m + n \times p + g = m + np + g$. So, to multiply $a + b + c$, 7, $m + n$, and $x^2 - y^2$ together, we should indicate the product thus, $7(a + b + c)(m + n)(x^2 - y^2)$, it being generally preferable that the monomial factors should precede the polynomial. So, again, $(x + y)(x + y) = (x + y)^2 = x^2 + 2xy + y^2$.

In performing the multiplication thus indicated, we may put the indicated products equal to the developed, or we may multiply as in the last section. Thus, $(a + b + c)(m - n) = (a + b + c)m - (a + b + c)n = am + bm + cm - an - bn - cn$.

In the same manner, indicate and then perform the multiplication in the following examples:

1. Multiply $ab + x - 1 + a$ by xyz.
2. Multiply $bcx - 16$ by $ac + mz$.
3. Multiply $5x + b$ by $3x - 3$ and by $5z^3$.
4. Multiply $x + 2xy + y$ by $x - y$.
5. Multiply $5 + b^2 - e$ by $a^2 - m$.
6. Multiply $13a^2y - 12x$ by $y - 1$.
7. Multiply $m^2 - n + z$ by $4a^2$ and by $5mn - 2$.
8. Multiply $3y^2$ by $x^3 - y^3$, by $2b$, by $y^2 + 4$.
9. Multiply $a + b$ by $a + b$ and by $a + b$.
10. Multiply $a - b$ by $a - b$ and by $a - b$.

The following examples are of frequent occurrence, and the results should be committed to memory.

Multiply $a + b$ by $a - b$.

$$\begin{array}{lll}
a & + b & \\
a & - b & \\
\hline
a^2 & + ab & \\
 & - ab & - b^2 \\
\hline
a^2 & & - b^2
\end{array}$$

Now $a + b$ is the sum of the quantities (53) a and b, and $a - b$ is the difference (53) of the same quantities; their product $a^2 - b^2$ equals the difference of the second powers of the same quantities. Hence,

82. *The sum of two quantities, multiplied by their difference, gives the difference of their second powers.*

Thus, $(2a^3 + 5bx)(2a^3 - 5bx) = 4a^6 - 25b^2x^2$.

Indicate and then perform the multiplication in the following examples, by the preceding rule:

11. $c + d$ multiplied by $c - d$.
12. $a + 12$ multiplied by $a - 12$.
13. $m - 1$ multiplied by $m + 1$.
14. $a^2 + 1$ multiplied by $a^2 - 1$.
15. $3m + 5a$ multiplied by $3m - 5a$.
16. $2ax^2 + 5y$ multiplied by $2ax^2 - 5y$.
17. $4b + 6m^2$ multiplied by $4b - 6m^2$.

Multiply $a + b$ by $a + b$.

$$\begin{array}{l} a + b \\ \underline{a + b} \\ a^2 + ab \\ \underline{\qquad\; ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

$a + b$ is the sum of [illegible] two quantities a and b, and $(a + b)(a + b)$, or $(a + b)^2$, is the second power or square of the same quantities. But $(a + b)^2 = a^2 + 2ab + b^2$; that is, it is equal to the second power or square of a, + twice the product of a and b + the square of b. Hence,

83. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

Thus, $(2a + 3b^2)^2 = 4a^2 + 12ab^2 + 9b^4$.

In the same way perform the following examples:

18. $(x + y)^2$. $x^2 + 2xy + y^2$
19. $(m + 2n)^2$.
20. $(a + 1)^2$. $a^2 + 2a + 1$
21. $(1 + m^2)^2$.
22. $(4a + 6)^2$.
23. $(bx + y)^2$.
24. $(5x^2 + y^3)^2$.
25. $(m^2a + 7)^2$.
26. $(2x^5 + 10z)^2$.
27. $(6p^2y + 3an^3)^2$.

What is the square of $a - b$?

$$\begin{array}{l} a - b \\ a - b \\ \hline a^2 - ab \\ \quad - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$

We see, then, that

84. *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

Thus, $(5ax^3 - 7p)^2 = 25a^2x^6 - 70apx^3 + 49p^2$.

It is generally easier to multiply twice the first term by the second; thus, twice $5ax^3 = 10ax^3$, and $10ax^3 \times 7p = 70apx^3$.

In the same way perform the following examples:

28. $(x - y)^2$. $x^2 - 2xy + y^2$
29. $(m - n)^2$. $m^2 - 2mn + n^2$
30. $(2a - p)^2$. $a^2 - 4ap + p^2$
31. $(xy - 8)^2$. $y^2 - 16xy + 64$
32. $(a - 1)^2$. $a^2 - 2a + 1$
33. $(1 - m^2)^2$. $2m^4 + m^4$
34. $(5m^3 - 3y)^2$. $23m^6 - 30m^3$ [illegible]
35. $(2ab - 4a)^2$. [illegible]
36. $(m^3x - 9z)^2$. m^6y [illegible]
37. $(7ab^2 - 12)^2$. [illegible] $64 - 16$

SECTION VIII.

DIVISION OF MONOMIALS.

85. DIVISION, *in Algebra, consists in finding how many times a given quantity, called the divisor, is contained in another given quantity called the dividend.*

The quantity thus found is called the *quotient.*

Division is the reverse of multiplication. It consists in separating a given product into two factors, one of which is given, and the other required. Thus, to divide 63 by 7, we separate 63 into two factors, one of which is 7, and the other must be 9, since $7 \times 9 = 63$. So $5ab \div a$, or $\frac{5ab}{a}, = 5b$, because $5b \times a = 5ab$. Again, $15a^3m^2n \div 3m^2n = 5a^3$, since $3m^2n \times 5a^3 = 15a^3m^2n$.. That is, we drop from the dividend all the prime factors (72) of the divisor, and write the remaining factors as the quotient.

To divide a^5 by a^2, we have $\frac{a^5}{a^2} = \frac{aaaaa}{aa}$. Now dropping the common factors, aa, we shall have $aaa = a^3$ for the quotient. Hence, when the dividend and divisor contain powers of the same letter, the exponent of that letter in the quotient will be found by subtracting the exponent of the divisor from that of the dividend. Thus, $a^3m^5n^3 \div a^2m^3n = am^2n^2$.

If any quantity be divided by itself, as a by a, am^3 by am^3, &c., the quotient is simply 1.

To determine the sign of the quotients, we have only to observe that the quotient multiplied by the divisor should reproduce the dividend with its sign. For instance, $+ ab = + a \times + b$, or $- a \times - b$. Hence $\frac{+ ab}{+ a} = + b$, and $\frac{+ ab}{- a} = - b$. If these quotients were otherwise, if the first, for example, were $- b$, then the product of $- b$ by $+ a$ would equal $- ab$ (see 80), which is not the given dividend.

As $- ab = - a \times + b$, or $+ a \times - b$, it follows that $\frac{- ab}{- a} = + b$, and $\frac{- ab}{+ a} = - b$.

Examining these results, we find that $\frac{+ ab}{+ a} = + b$, and $\frac{- ab}{- a} = + b$; hence, *like* signs in the divisor and dividend give + in the quotient. And since $\frac{- ab}{+ a} = - b$, and $\frac{+ ab}{- a} = - b$, it follows that unlike signs give — in the quotient.

Hence, we have the following rule for the division of monomials.:

86. (1.) *Divide the coefficient of the dividend by that of the divisor, and annex to the result in their proper order all letters found in the dividend and not in the divisor; also all letters common to both, giving such letters an exponent found by subtracting the exponent of the divisor from that of the dividend. If the exponents are equal, omit the letters in the quotient.*

(2.) *If the signs in the dividend and in the divisor are* ALIKE, *the quotient must have the sign* $+$; *if* UNLIKE, *the sign* $-$.

Thus, $36a^3bm^2xy$ divided by $-9amxy = -4a^2bm$.

1. Divide $18abmx^2$ by $9abx^2$. - 2m
2. Divide $27abc^3mx$ by $3bcx$. 9ac
3. Divide abx by abx.
4. Divide $39az$ by $-13z$. - 3
5. Divide $108m^5n^4x$ by $-12m^2x$. — 9mn³ 4
6. Divide $36a^2x^2y^2$ by $-9a^2x^2y^2$. - 4
7. Divide $-8a^2z^3$ by $8a^2z^3$. - 1
8. Divide $-12am^3n^5$ by $4m^2n^2$. 3
9. Divide $-28a^7d^6m^5n^4p^3$ by $7a^2m^5p^3$. 4a 4,
10. Divide $81abc^2$ by $9abc^2$. + 9
11. Divide $-81abc^2$ by $-9abc^2$. +
12. Divide $-xyz^2$ by $-xyz^2$. + 1
13. Divide xyz^2 by $-xyz^2$. — 1
14. Divide $-32ac^2dy^3$ by $-4ay^3$.
15. Divide $-27ab^2xy^3z$ by $9abx$. —
16. Divide a^6 by a^3; by $-a^3$.
17. Divide $-a^3d^4$ by ad^3; by $-ad^3$. and
18. Divide $16a^{10}d^4x^2$ by $-4a^5d^3x^2$.
19. Divide $-39a^3m^3y^4$ by $13a^2m^2y^3$. — 7
20. Divide $-40mnpy^2$ by $-5m$.

87. *Any algebraic quantity is*

(1.) DIVISIBLE, *when it contains all the prime factors of the quantity by which it is divided.*

(2.) PARTIALLY DIVISIBLE, *when it has prime factors in common with some of those in its divisor.*

(3.) INDIVISIBLE, *when it has no prime factor found in its divisor.*

Thus, $15a^2bc$ is divisible by $5ac$, since the factors 5, a, and c are contained in $15a^2bc$. Cancelling the common factors, we have $3ab$ for the quotient.

Again, $21a^3bc$ is partially divisible by $7a^2mn$, for the factors 7 and a^2 are common to both. Hence, cancelling the common factors, as above, we shall have $\frac{21a^3bc}{7a^2mn} = \frac{3abc}{mn}$.

But $39a^2m$ is not divisible by $14p^2y$, since $39a^2m$ and $14p^2y$ have no factor in common. The division may be indicated by a fraction; thus, $\frac{39a^2m}{14p^2y}$.

88. *Two quantities, having no factors in common, are said to be* PRIME *to each other.*

Thus, $39a^2m$ and $14p^2y$, in the last example, are prime to each other.

In the following examples, indicate the quotient in the form of a fraction; which must be reduced, if possible, by striking out the common factors in the numerator and denominator.

Thus, a divided by $-ax = \frac{a}{-ax} = -\frac{1}{x}$ (86 (2)).

21. Divide $16ab$ by $4x$. $\frac{4ab}{x}$
22. Divide $15xy^2$ by $5axy$.
23. Divide $5ad$ by $3x$. $\frac{5ad}{3x}$
24. Divide $-5a^5$ by $5a^4$; by $-5a^7$.
25. Divide abm^3 by $4abm^3$.
26. Divide a by $-b$; by $+b$.

27. Divide $-9ax^2$ by $4x^2$; by $-4x^2$. $-\frac{9a}{4}$, $\frac{9a}{4}$

28. Divide $-12b^2cd$ by $-3bcx$.

29. Divide $-7ab^2$ by $-5a$; by $5a$.

30. Divide $7ab^2$ by $-5a$; by $5a$. $\frac{7b}{5}$

31. Divide ab by abc^2. $\frac{1}{c^2}$

32. Divide $5m^2n^2$ by $5abm^2n^2$. $\frac{1}{ab}$

33. Divide $18ax$ by $-6xy^5$.

34. Divide ab by $3cd$. $\frac{ab}{3cd}$

35. Divide $-10a^5$ by $10a^5b^2x^3$.

Remark.—The relation of the dividend, divisor, and quotient to each other, is such that

(1.) The product of the *divisor* and *quotient* is always equal to the *dividend.*

(2.) If the dividend alone be multiplied, or the divisor divided by any number, the *quotient* will be *multiplied* by the same number. Thus, $18 \div 6 = 3$; but $(18 \times 2) \div 6 = 3 \times 2$; and $18 \div (6 \div 2) = 3 \times 2$ also.

(3.) If the divisor alone be multiplied, or the dividend divided by any number, the *quotient* will be *divided* by the same number. Thus, $24 \div 3 = 8$; but $24 \div (3 \times 2) = 8 \div 2$; and $(24 \div 2) \div 3 = 8 \div 2$.

(4.) If the divisor and the dividend *both* be multiplied by the same number, the *quotient* will be unchanged. Thus, $12 \div 4 = 3$, and $(12 \times 5) \div (4 \times 5) = 3$ also.

(5.) If the divisor and the dividend *both* be divided by the same number, the quotient will be unchanged. Thus, $20 \div 10 = 2$, and $(20 \div 5) \div (10 \div 5) = 2$ also.

SECTION IX.

DIVISION OF POLYNOMIALS.

When the divisor or quotient is a *monomial.*

If we have as a dividend, the product of $a + b - c$ by d, that is, $(a + b - c)d$, and the factor d be given as a divisor, the quotient must be the other factor $a + b - c$. On the contrary, if $a + b - c$ be given as the divisor, the quotient must be d.

If, instead of the indicated product $(a + b - c)d$ (see Sec. VI.), we have the developed product $ad + bd - cd$, a mere inspection of the terms will show us that the monomial factor is found as a factor in each term. Hence, to divide a polynomial by a monomial,

89. *Divide each term of the dividend by the divisor, as in the division of monomials.*

Thus, $18a^3x + 15a^2 - 21ab$ divided by $-3a$, will give for the quotient, $-6a^2x - 5a + 7b$.

1. Divide $3abc + 12ab^2x - 9a^2b$ by $3ab$. $c + 4b$[illegible] $- 3a$
2. Divide $10ax^5 - 15x^3$ by $-5x^2$. $-2ax^3 + 3x$
3. Divide $a^2d^2x^2 + ax - a^3dx^3$ by ax. [illegible] $+ 1 - a^2$[illegible]
4. Divide $a^2d - a + ad$ by a. $ad - 1 + d$
5. Divide $6ab + 12ac - 18ad$ by $-6a$. b [illegible]
6. Divide $16a - 8 + 12y^2 - 20ady + 4m$ by 4. $4a$ [illegible]
7. Divide $-28abm - 14b^2m^2 - 49b^2mx$ by $7bm$. $-4a$ [illegible]
8. Divide $-16ab - 8a$ by $-8a$. $2b + 1$
9. Divide $-12x^5y^2 + 18x^3y$ by $-6x^2$. [illegible]
10. Divide $27a^2d - 9ad$ by $9ad$. $3a - 1$

If, instead of the monomial factor, the polynomial should be the divisor, since the former is a factor in every term of the dividend, we have only to divide any term of the dividend by the corresponding term of the divisor.

Thus, if $a + b - c$ be the divisor, and $ad + bd - cd$ be the dividend, we may divide either ad by a, bd by b, or cd by c, and in each case obtain the same quotient d. It will, however, be sufficient to divide the *first* term of the dividend by the first term of the divisor; and then prove the quotient to be correct by (88, Rem. 1). Hence, we have the following rule:

90. *Divide the first term of the dividend by the first term of the divisor. The quotient will be the other factor of the dividend.*

Remark.—The terms of the dividend and of the divisor must be arranged in the same order.

$3a^2 - 2ab$ is divided by $3a - 2b$ as follows:

$3a^2 \div 3a = a$; and since $(3a - 2b)a = 3a^2 - 2ab$, a is the required quotient.

11. Divide $12a^2 + 6ab$ by $4a + 2b$. $3a$
12. Divide $18a^3 + 6a^2b - 12a^2x$ by $6a + 2b - 4x$. $3a^2$
13. Divide $14ab^2 + 21ac + 49ax$ by $2b^2 + 3c + 7x$.) a
14. Divide $-10a^2 + 5ax$ by $2a - x$. $-5a$
[illegible]. Divide $3a^2 - 2cd$ by $-3a^2 + 2cd$. -1
16. Divide $ab^3 - b^3c$ by $a - c$.

When the given polynomial divisor is not exactly contained in the dividend, we must indicate the division in the form of a fraction, as in the case of monomials. See examples under (88).

Divide $a + b$ by $c - d$. *Ans.* $\dfrac{a+b}{c-d}$.

17. Divide $8a + b - c$ by $m + 5$. $\dfrac{8a+b-c}{m+5}$
18. Divide $17xy - m + 7$ by $a + 10$. $\dfrac{17xy-m+7}{a+10}$
19. Divide $a - 7 + m$ by $a + 5$. $a - 7 + m$
20. Divide $x^2 - y^2$ by $m^2 + n^2$. $\dfrac{x^2-y^2}{m^2+n^2}$

SECTION X.

DIVISION OF POLYNOMIALS.

When the divisor and quotient are both *polynomials*.

If we have, as a dividend, the product of $a + b$ by $c + d$, that is, $(a + b)(c + d)$, and the factor $a + b$ be given as the divisor, the quotient must be $c + d$; or, if $c + d$ be given as a divisor, the quotient must be $a + b$.

Now, if the partial products were given, and one of the factors, as $a + b$, we should have $(a + b)c + (a + b)d$. Dividing each term by $(a + b)$ as a monomial, we have the quotient $c + d$.

Again, if the developed product were given, the case would be no more difficult; for $(a + b)c$ and $(a + b)d$ would give $ac + bc + ad + bd$; and, by inspection, we see that c is a monomial factor in the first two terms, and d in the second two. Hence, dividing the first two by $a + b$ (90), we get c, and the second two by the same, we have d that is, $c + d$. This division may be represented thus,

$$\begin{array}{llll}
ac + bc + ad + bd & \multicolumn{1}{|l}{a + b} & \text{Divisor.} \\
\cline{2-2}
\underline{ac + bc \qquad\qquad\quad} & \quad c + d & \text{Quotient.} \\
\;0 \quad\;\; 0 & & \\
\qquad\qquad ad + bd & & \\
\qquad\qquad \underline{ad + bd} & &
\end{array}$$

By (90), we divide the first term of the dividend by the first term of the divisor, writing c in the quotient; then multiplying the divisor by c, we obtain $(a + b)c = ac + bc$, the first partial product. This subtracted by (70) from the dividend, leaves $ad + bd$, the second partial product. Dividing again, as in (90), we get d, and multiplying and subtracting, as before, we exhaust the dividend.

When the developed product results from various reductions of similar terms, the case is more difficult. Thus, to divide $a^2 - b^2$ by $a + b$, we know (82) that the other factor or quotient is $a - b$, and that the dividend is $(a + b)(a - b) = (a + b)a - (a + b)b = a^2 + ab - ab - b^2 =$ by (66) $a^2 - b^2$. If we had kept the forms $(a + b)a - (a + b)b$, or $a^2 + ab$ [illegible] $b - b^2$, the case would not differ from the preceding. Thus,

$$\begin{array}{lcllll}
\textit{No. } 1. & & \textit{No. } 2. & & & \\
a^2 + ab - ab - b^2 & = & a^2 - b^2 & \multicolumn{1}{|l}{a + b} & \text{Divisor.} \\
\cline{4-4}
\underline{a^2 + ab} & = & \underline{a^2 + ab} & a - b & \text{Quotient.} \\
\;0 \quad\; 0 & & 0 & & \\
\quad\;\; \overline{-ab - b^2} & = & -ab - b^2 & & \\
\quad\;\; \underline{-ab - b^2} & & \underline{-ab - b^2} & &
\end{array}$$

In No. 1, $a^2 + ab$ is the first partial product; in No. 2, ab is dropped by (66). But as the first term, a^2, remains in both, we have only to divide this (90) by the first term of the divisor. Placing the quotient a as the first term of the quotient, we multiply the entire divisor by it, and thus obtain the whole partial product $(a + b)a = a^2 + ab$, the last term of which was lost in reduction. Subtracting this by (70) from $a^2 - b^2$, we have $a^2 - b^2 - a^2 - ab =$ (66) $- b^2 - ab$, the same in No. 2 as in No. 1, except the order of the terms is changed. To preserve the same order as in the divisor, the term containing a should be placed first, thus, $- ab - b^2$. Dividing again by a, the first term of the divisor (90), we have $- b$. Multiplying and subtracting as before, we exhaust the dividend.

From these illustrations, we have the following rule for the division of polynomials:

91. (1.) *Arrange the divisor and dividend alike, that is, according to the powers of some letter.*

(2.) *Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.*

(3.) *Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend; the remainder, arranged like the divisor, will form a new dividend.*

(4.) *Divide the first term of the new dividend by the first term of the divisor; the result will be the second term of the quotient.*

(5.) *Multiply the whole divisor by the second term of the quotient, and subtract the product from the last dividend for a new dividend, and so continue till there is no remainder.*

REMARK.—As, in every case, we simply divide a monomial by a monomial, the rule for signs in (86) must be observed. In subtracting the partial products, change the signs, as in (70).

Thus, $- 3ax^2 - 3a^2x + a^3 + x^3$ is divided by $x + a$ as follows:

$$\begin{array}{rrrrl}
a^3 & -\,3a^2x & -\,3ax^2 & +\,x^3 & \left\{\begin{array}{l} a+x \\ \hline a^2-4ax+x^2 \end{array}\right. \\
a^3 & +\;\;a^2x & & & \\ \hline
 & -\,4a^2x & -\,3ax^2 & +\,x^3 & \\
 & -\,4a^2x & -\,4ax^2 & & \\ \hline
 & & +\;\;ax^2 & +\,x^3 & \\
 & & +\;\;ax^2 & +\,x^3 & \\ \hline
 & & & 0 &
\end{array}$$

In this example, we first arrange the dividend and divisor according to the powers of a. The divisor is placed at the right of the dividend, and the quotient, for convenience, beneath it. Dividing a^3 by a, we have a^2 for the first term of the quotient. We then subtract $(a + x)a^2$ from the dividend, which gives $-4a^2x$ for the first term of the new dividend. Dividing, as before, by a, we have $-4ax$ for the second term of the quotient; and subtracting $-4ax\,(a + x)$, we obtain $+ax^2$ for the first term of the last dividend; dividing this by a, we have x^2 for the last term of the quotient, since $(a + x)x^2$ is equal to the last dividend.

$b^3 + c^3$ is thus divided by $b + c$:

$$\begin{array}{rrrl}
b^3 & +\,c^3 & & \left\{\begin{array}{l} b+c \\ \hline b^2-bc+c^2 \end{array}\right. \\
b^3 & +\,b^2c & & \\ \hline
 & -\,b^2c & +\,c^3 & \\
 & -\,b^2c & -\,bc^2 & \\ \hline
 & & +\,bc^2+c^3 & \\
 & & +\,bc^2+c^3 & \\ \hline
 & & 0 &
\end{array}$$

1. Divide $a^2 + ab + ac + 5a + 5b + 5c$ by $a + b + c$. a + 5
2. Divide $a^2 - 6a + ab - ac + 6c - bc$ by $a - c$. 6
3. Divide $-3m^2xy - 2m^2x + 6m^3x + mxy + 2mx - xy$ by $2m - y$.
4. Divide $x^2 + x^3 + x^2y + xy + 3xz + 3z$ by $x + 1$.
5. Divide $a + b - c - ax - bx + cx$ by $a + b - c$.

7

6. Divide $a^3 - 3a^2y - y^3 + 3ay^2$ by $a - y$.

7. Divide $4a^3 + 3acm + 4a^2c + 3c^2m - 4a^2m - 3cm^2$ by $a + c - m$.

8. Divide $5m^3 - 2myz - 5m^2n + 2nyz + 5m^2z - 2yz^2$ by $m - n + z$.

9. Divide $x^4 - y^4$ by $x - y$. $x^3 +$

10. Divide $a^4 - b^4$ by $a + b$.

11. Divide $m^5 + n^5$ by $m + n$.

12. Divide $x^3 - 1$ by $x - 1$.

13. Divide $1 + x^5$ by $1 + x$.

14. Divide $2a^2b^2c + 3ab^3 - ab^2 + b^3c + b^2x - 3a^2b - 2a^3c + a^3 - abc - ax$ by $3ab + 2a^2c - a + bc + x$.

15. Divide $-16a^2my - 3a^2dy + 64am^2 + 12adm$ by $16am + 3ad$.

16. Divide $12a^2b - 8a - 144ab + 96$ by $a - 12$.

17. Divide $3a^5 - 33a^3b^2 + 14a^2b^3 + 16a^4b$ by $a^2 + 7ab$.

18. Divide $x^3 + 9x^2 + 4x - 80$ by $x + 5$.

19. Divide $c^8 - 16d^8$ by $c^2 - 2d^2$.

20. Divide $a^2x - b^2x + 8x - a^2y^3 + b^2y^3 - 8y^3$ by $x - y^3$.

$a^2 - b^2 + 8$

SECTION XI.

SEPARATING COMPOSITE QUANTITIES INTO FACTORS.

The prime factors (72) of a product are easily ascertained when the multiplication is only indicated; but for the most part, they disappear when the multiplication is performed.

Thus, in $3 \times 5 \times 7$, the factors 3, 5, and 7 are at once seen; so in abc, a^3m^2p. But in 105, or in $a^3 - 1$, little or no trace of the factors can be discovered. It is often necessary to return from a developed to an indicated product, that is, to separate a product into its factors.

(*a.*) *Monomials.* In monomials composed of letters alone, we can have only an indicated product, and the prime factors are readily discovered by mere inspection. But in monomials containing coefficients, it is often necessary to resolve the coefficient into its prime factors. Thus, in $36a^2m^3y$, if we divide successively by the prime factors 2 and 3, we shall have $36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$. Then $36a^2m^3y = 2 \times 2 \times 3 \times 3 \times a \times a \times m \times m \times m \times y = 2^2 \times 3^2 \times a^2 \times m^3 \times y$. Hence, ~~to~~ resolve a monomial,

92. *Separate the coefficient into its prime factors, and after these, write in order the letters, giving each its proper exponent.*

NOTE.—The learner should be careful to separate the coefficient into *prime* factors. Thus, 4 × 9, 3 × 12, 2 × 18, and 6 × 6, are all factors of 36, but 2 and 3 are the only prime factors. To ascertain the prime factors, begin, as in arithmetic, and divide successively by the prime factors 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, &c., repeating the division by each, when possible, till the given number is resolved.

Thus, to separate $520a^3bc^2$ into its prime factors, we have

$$\begin{array}{r|l}
2 & 520 \\
2 & 260 \\
2 & 130 \\
5 & 65 \\
13 & 13 \\
\hline
 & 1
\end{array}$$

Here $2 \times 2 \times 2 \times 5 \times 13 = 2^3 \times 5 \times 13 = 520$.

The prime factors then are $2^3 \times 5 \times 13 \times a^3 \times b \times c^2$.

(× 2 × 3

Separate the following monomials into their prime factors:

1. $81a^4b$. 4 × a 4 ×
2. $24m^2n$. 3 × 3 ×
3. $52xy^2z$.
4. $72a^3m^2n^5$.
5. $96ax$.
6. $104b^2cd^3$.
7. $121d^4x^2$.
8. $637y^8z^9$.
9. $625ap^2x$.
10. $130m^3nx^2y$.

(*b.*) *Binomials.* Most binomials are inseparable; as $a + b$, $ax + bc$. But, in general,

93. *Any binomial may be separated into factors,*

(1.) *When both its terms contain a common monomial factor,—*

(2.) *When it is the* DIFFERENCE *of* ANY *similar powers, or the* SUM *of similar* ODD *powers of two quantities.*

(1.) $9a^2m + 18am^2 = 9am(a + 2m) = 3^2am(a + 2m)$. Here we divide each term by the greatest monomial factor $9am$, which is readily ascertained by an inspection of the terms. The other factor is a binomial $a + 2m$, which must be included in a parenthesis.

(2.) $a^2 - b^2 =$ (82) $(a + b)(a - b)$; and $(a^2 - 1) = (a + 1)(a - 1)$, since the 1 in $(a^2 - 1) = 1^2$. It must be remembered that all powers and all roots of 1 are 1.

$m^3 - m$ by (93 (1)) $= m(m^2 - 1) = m(m - 1)(m + 1)$. Again, $a^3 - b^3$ will be found, by trial, equal to $(a - b)(a^2 + ab + b^2)$; and $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$; and, in general,

94. *The difference of any similar powers is divisible by the difference of the quantities; and the sum of similar odd powers by the sum of the quantities.*

Thus, $a^5 - y^5$, $a^4 - y^4$, &c., are each divisible by $a - y$, while $a^3 + y^3$, $a^5 + y^5$, $a^7 + y^7$, &c., are each divisible by $a + y$.

The binomial $a^4 - b^4 =$ (82) $(a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$, since $a^2 - b^2 = (a + b)(a - b)$. So $m^5 - m = m(m^4 - 1) = m(m^2 + 1)(m^2 - 1) = m(m^2 + 1)(m + 1)(m - 1)$. $a^6 - b^6 =$ (82) $(a^3 + b^3)(a^3 - b^3) =$ (94) $(a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2)$. $d^6 + d = d(d^5 + 1) =$ (94) $d(d + 1)(d^4 - d^3 + d^2 - d + 1)$.

(*c.*) *Trinomials.* Although most trinomials are prime quantities, as $a + b + c$, $m^2 + 2mx - n^2$, yet

95. *Any trinomial may be separated into factors,*

(1.) *When each term contains a common monomial factor*, or

(2.) *When its extreme terms are squares of two different monomials, and the middle term is twice their product.*

Remark.—In this case, the extreme terms must both be +, but the middle term may be either + or —. See (83) and (84).

Thus, $a^2x - mx^2 + xyz = x(a^2 - mx + yz)$. So $6a^2 + 12ab + 6b^2 = 6(a^2 + 2ab + b^2) = (95, (2))\ 6(a + b)^2$, and $mx^2 - 2mxy + my^2 = m(x^2 - 2xy + y^2) = m(x - y)^2$.

(*d.*) *Any polynomial.*

96. *Any polynomial may be separated into factors,—*

(1.) *When each term contains a monomial factor,—*

(2.) *When it is any power of a binomial.*

Thus, $a^2m^2p - 3a^3mn + 5a^4mn^2 - a^2m = a^2m\,(mp - 3an + 5a^2n^2 - 1)$; and $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$. Also $x^3 - 3x^2y + 3xy^2 - y^3 = (x - y)^3$.

Frequently, a polynomial may be separated into factors by a combination of the foregoing methods.

Thus, $b^2 + 2bc + c^2 - a^2 = (94)\ (b + c)^2 - a^2$. Now regarding $(b + c)^2$ as the square of one term, and a^2 as that of another, we shall have, by (82), $(b + c + a)(b + c - a)$.

Take the polynomial $m^2 - p^2 + 2p - 1$, the last three terms would be the square of $p - 1$ but for the signs. (See 95, Rem.) But by (69, Rem.), these terms may be made a subtrahend in a parenthesis, with their signs changed; thus, $m^2 - (p^2 - 2p + 1) = m^2 - (p - 1)^2 = (82)\ (m + p - 1)(m - p + 1)$.

Separate the following quantities into their prime factors:

11. $15a^2 + 25ab - 5ax$.
12. $12x^2 + 24xy + 12y^2$.
13. $5x^2 + 10xy + 5x^2y^2$.
14. $52a^2b^2c^2 - 39abc^2$.
15. $36x^6 - 9y^2$. See (82).
16. $40x^4 - 40y^4$.
17. $25x^2 - 9y^2$.
18. $9x^3 - 9$.
19. $15x^2 - 15y^2$.
20. $52x^6 - 52y^6$.
21. $44x^5 + 44y^5$.
22. $9x^2 - 18x + 9$.

23. $108m^2n - 96mn^2 + 60m$.

24. $35m^5x - 25m^2x^2 + 3m$.

25. $14a^2z + 7zy - 17xz$.

26. $27m^5 - 36m^3n + 54m^3ny - 18m^3$.

27. $10x^3 - 30x^2y + 30xy^2 - 10y^3$.

28. $4a^3 + 12a^2b + 12ab^2 + 4b^3$.

29. $c^2 - 2cd + d^2 - x^2$. See (96).

30. $a^2 - m^2 + 2m - 1$.

If the product of the factors of the dividend is simply indicated, as $3^2 \times 5^2 \times a^2$, it may be divided by any one of its factors by simply dropping that factor. Thus, to divide $3^2 \times 5^2 \times a^2$ by $3^2 \times 5 \times a$, we should have $1 \times 5 \times a = 5a$ for the quotient. The result is the same as if the actual product $225a^2$ were divided by $45a$.

In the same way, $2^4 \times 17ad^2(x^2 - y^2) \div 2^2 \times 17d(x + y) = 2^2 \times ad(x - y) = 4ad(x - y)$, the indicated quotient.

It is often convenient to separate the dividend and divisor into factors, and then proceed to divide as above.

Note.—To divide $(a + b)^5$ by $(a + b)^3$, we may consider the quantity within the parenthesis as a monomial, and then divide, as in (86), by subtracting the exponent of the divisor from that of the dividend. Thus, $(a + b)^5 \div (a + b)^3 = (a + b)^2$.

In the same way perform the following examples:

31. Divide $15a(x^2 - y^2)$ by $5(x - y)$.

32. Divide $14m^3(a + b)^3 (m^2 - n^2)$ by $7m^2(a + b)(m + n)$.

33. Divide $14(x^3 - y^3)(x^3 + y^3)$ by $7(x^3 - y^3)(x + y)$.

34. Divide $20bc(m^6 - 81)x^3(x + y)^2$ by $5bx(m^3 + 9)(x + y)$.

35. Divide $45abx^2(16m^2 - 36y^2)(a - b)^2$ by $(4m + 6y)(a - b)9ax$.

36. Divide $(a + b)^5 (a - y)^3 (a - x)^2$ by $(a + b)^3 (a - y)^2 (a - x)$.

37. Divide $a^4(m^2 - 2mn + n^2)$ by $a^2(m - n)$.

38. Divide $36a^2x^2 + 72abx^2 + 36b^2x^2$ by $18x(a + b)$.

39. Divide $(54bcdm - 54bcd)(x^4 - 25)$ by $27bc(x^2 - 5)(m - 1)$. $= 2d(x^2+5)$

40. Divide $14x^6 - 14y^6$ by $7(x^2 - y^2)(x + y)$.

$= 2x^3+y^3)(x-$ [illegible] 2

SECTION XII.

MULTIPLES AND DIVISORS.

The successive products arising from multiplying any quantity by 1, 2, 3, 4, 5, 6, 7, &c., indefinitely, are called *multiples* of that quantity. Thus, a, $2a$, $3a$, $4a$, $5a$, ba, are all multiples of a. Hence,

96. *A* MULTIPLE *of a quantity is any product which contains it as a factor.*

A product containing two or more factors, is as much a multiple of one as of the other. Hence, it is common to them all as a multiple. That is,

97. *A* COMMON MULTIPLE *of two or more quantities is a product which contains them all as factors.*

Thus, a^2bc is a common multiple of a, a^2, ab, a^2b, ac, a^2c, abc, and a^2bc. There may be any number of common multiples of these quantities, as $3a^2bc$, a^2bcx, a^2bcdm, &c., but a^2bc is the *least* common multiple, since it is the smallest quantity which will contain them all. So 24 is a common multiple of 2, 3, 4, 6, and 12, but 12 is the *least* common multiple of all these numbers. Hence,

98. *The* LEAST COMMON MULTIPLE *of two or more quantities is the least product which contains them all as factors.*

To find the least common multiple of two numbers, as 45 and 54, we separate them into their prime factors, $45 = 3^2 \times 5$ and $54 = 2 \times 3^3$; therefore $5 \times 2 \times 3^3$ will contain both of the given quantities, since it contains all their prime factors. We reject 3^2, since we have a higher power of the same quantity, namely, 3^3, which, of course, contains 3^2. Hence, to find the least common multiple of several quantities,

99. *Resolve each of the given quantities into its prime factors; and then take the product of all the different prime factors, giving each the highest power found in any of the given quantities.*

The least common multiple of $9ax^2$, $a+b$, and $18a^3x$, is thus found: $9ax^2 = 3^2 \times a \times x^2$; $a+b=(a+b)$; $18a^3x = 3^2 \times 2 \times a^3 \times x$; hence, $3^2 \times 2 \times a^3 \times x^2 \times (a+b) = 18a^3x^2(a+b) = 18a^4x^2 + 18a^3bx$, the least common multiple sought.

Find the least common multiple of the following quantities:

1. $7a^2d$, $14abd^3$, and $28b^2$.
2. $12mn$, $20m^3$, $15x$, and $3n^2$.
3. $8x^2$, $7xy$, $4y^2$, and $14mx$.
4. $9m^3$, $15a$, and $m+n$.
5. a^2-b^2, $a+b$, and $a-b$.
6. x^2+y^2, $x+y$, $x-y$, and $5a^2$.
7. $5a$, $7(x+y)$, and $3mn^2$.
8. $8a$, $a+b$, and $9a^2m$.

DIVISORS.

A *divisor* of a quantity is any quantity that will exactly divide it; thus, 5 is a divisor of 15, a is a divisor of ab.

A *common divisor* of two or more quantities is a quantity which will exactly divide them all; thus, 3 is a common divisor of 9, 36, and 45, a is a common divisor of ab, abc, and abd.

The *greatest common divisor* of two or more quantities is the greatest quantity that will divide them all. Thus, 9 is the *greatest* common divisor of 9, 36, and 45; and ab of ab, abc, and abd.

To find the greatest common divisor,

100. *Separate the given quantities into their prime factors, and then take the product of the factors common to all the quantities.*

To find the greatest common divisor of $8a^3$, $12ab$, $20a^2x$, and $40ad^2$, we must separate the quantities into their prime factors, thus:

$$\begin{aligned} 8a^3 &= 2^3 \times a^3. \\ 12ab &= 2^2 \times 3 \times a \times b. \\ 20a^2x &= 2^2 \times 5 \times a^2 \times x. \\ 40ad^2 &= 2^3 \times 5 \times a \times d^2. \end{aligned}$$

Now taking the product of all the common factors, we have $2^2 \times a = 4a$, the greatest common divisor.

The greatest common divisor of $m^2 + 2mn + n^2$, and $m^2 - n^2$, is found thus:

$$\begin{aligned} m^2 + 2mn + n^2 &= (m + n)^2. \\ m^2 - n^2 &= (m + n)(m - n). \end{aligned}$$

Hence, $m + n$ is the divisor sought.

Find the greatest common divisor of the following quantities:

9. $15a^2mn$, $25am^2$, and $30amn^3$.
10. $27x^2y$, $45ax^2$, $72a^2mx^2$, and $36x^2y^2$.
11. $a^2 - 2ab + b^2$ and $a - b$.
12. $m^2 - 1$, $m - 1$, and $m^2 - 2m + 1$.
13. $7(a + b)^5$ and $14m(a + b)^2$.
14. $x^3 + y^3$, $x^2 - y^2$, and $x^2 + 2xy + y^2$.
15. $15a^2bd - 30a^2dm$ and $6adm + 24bdm^2$.
16. $7a^2dx + 14adm^2x$ and $8am^3x - 12adx$.

CHAPTER III.

FRACTIONS.

SECTION I.

DEFINITIONS AND NATURE OF FRACTIONS.

101. *Algebraic quantities* (60) *may be either*

(1.) ENTIRE; *as*, 15, *am*, $p^2 + y^2$, *or*

(2.) FRACTIONAL; *as*, $\frac{3}{5}, \frac{a}{m}, \frac{p^2}{y^2}, \frac{a^2 - b^2}{m + n}$.

102. *Fractions, in Algebra, have the same meaning as in Arithmetic.*

Thus, $\frac{a}{b}$ means that 1 is divided into b equal parts, and then a of these parts are taken; so with $\frac{3}{5}, \frac{m}{ab}$.

103. *The* DENOMINATOR *shows into how many parts the unit is divided; as*, 4 *in* $\frac{3}{4}$, $m + n$ *in* $\frac{ab}{m + n}$.

104. *The* NUMERATOR *shows how many parts are taken; as*, 3 *in* $\frac{3}{4}$, ab *in* $\frac{ab}{m + n}$.

105. *The numerator and the denominator are called the* TERMS *of the fraction.*

106. *A* PROPER *fraction is one whose value is less than a unit; as*, $\frac{3}{7}, \frac{a}{a + b}, \frac{x - y}{x + y}$.

107. *An* IMPROPER *fraction is either equal to or greater than a unit; as*, $\frac{3}{3}, \frac{14}{5}, \frac{a}{a}, \frac{ab + m}{a}$.

108. *A* MIXED *quantity is an entire quantity connected with a fraction by the sign + or —, as,* $a + \frac{m}{x}$, $5d - \frac{x}{m - n}$.

109. *A fraction is an indicated quotient, the numerator representing the dividend, and the denominator the divisor.*

110. *The* VALUE *of a fraction is the quotient, whether indicated, or obtained by performing the division.* Hence, by (88, Rem.),

111. *Multiplying the numerator, or dividing the denominator,* MULTIPLIES *the value of the fraction.*

Thus, the value of $\frac{24}{6}$ is 4. If now we multiply the numerator by 2, we shall have $\frac{48}{6} = 8$, or *twice* the previous value of the fraction. Again, if we divide the denominator by 3, we shall have $\frac{24}{2} = 12$, or *three* times the original value of the fraction.

112. *Dividing the numerator, or multiplying the denominator,* DIVIDES *the value of the fraction.* See (88, Rem.)

Thus, the fraction $\frac{18}{3} = 6$. Dividing the numerator by 2, we have $\frac{9}{3} = 3$, or $6 \div 2$. So if we multiply the denominator by 2, we have $\frac{18}{6} = 3$. Here, also, the value of the original fraction, which was 6, is divided by 2.

$\frac{abc}{a} = bc$; but dividing the numerator by b, we have $\frac{ac}{a} = c$, which is the same as bc divided by b. So if we multiply the denominator by b, we shall have $\frac{abc}{ab} = c$, which is also equal to the previous value bc divided by b.

113. *Multiplying or dividing both terms of a fraction by the same quantity, does not alter its value.* See (88, Rem.)

Thus, $\frac{3}{4} = \frac{3 \times 5}{4 \times 5}$, or $\frac{15}{20}$, for it is the same as though we had multiplied $\frac{3}{4}$ by $\frac{5}{5} = 1$; so $\frac{9}{18} = \frac{9 \div 9}{18 \div 9} = \frac{1}{2}$, for it is the same as though we had divided $\frac{9}{18}$ by $\frac{9}{9}$, or 1.

SECTION II.

REDUCTIONS.

(*a.*) To reduce a fraction to an equivalent fraction having any required denominator,

114. *Divide the required denominator by that of the given fraction, and multiply both terms of the fraction by the quotient.* See (113).

Thus, to reduce $\frac{5}{7}$ to 28ths. We find that $28 \div 7 = 4$, hence we must multiply both terms by 4; thus, $\frac{5 \times 4}{7 \times 4} = \frac{20}{28}$. The *value* of the fraction is the same as before (64).

$\frac{x}{a+b}$ is thus changed to a fraction having $a^2 - b^2$ for its denominator. $a^2 - b^2 \div a + b = a - b$; multiplying both terms by $a - b$, we have $\frac{x(a-b)}{(a+b)(a-b)} = \frac{ax - bx}{a^2 - b^2}$.

1. Change $\frac{7a}{13}$ to a fraction having $39x$ for its denominator.

2. Change $\frac{a^2b^2}{5mn}$ to a fraction having $45m^3n$ for its denominator.

3. Change $\frac{a+b}{c}$ to a fraction having $7c^2$ for its denominator.

4. Change $\frac{8y}{a+b}$ to a fraction having $a^2+2ab+b^2$ for its denominator (95 (2)).

5. Change $\frac{x^2-y^2}{x^2+y^2}$ to a fraction having x^4-y^4 for its denominator (93 (2)).

(*b*.) To reduce an entire quantity, or integer, to an improper fraction of any required denominator,

115. *Consider the integer as a fraction with* 1 *for its denominator, and multiply both terms by the required denominator.*

Thus, $8=\frac{8}{1}$, and can be reduced to 7ths, as in (114), $\frac{8\times7}{1\times7}=\frac{56}{7}$. So ab can be reduced to a fraction having $a+b$ for its denominator. $\frac{ab}{1}=\frac{ab(a+b)}{1(a+b)}=\frac{a^2b+ab^2}{a+b}$

6. Change $8a$ to a fraction having $5b^2$ for its denominator.
7. Change $a+b$ to a fraction having c for its denominator.
8. Change $5x-by$ to a fraction with $4x$ for its denominator.
9. Change $x+b-5a$ to a fraction with $6y$ for its denominator.
10. Change $a+b$ to a fraction with $a-b$ for its denominator.
11. Change $2a^2b+d^3-5$ to a fraction with $5ab$ for its denominator.

(*c*.) To reduce a mixed quantity to an improper fraction,

116. *Change the entire quantity to a fraction whose denominator shall be the same with that of the annexed fraction; unite the numerators by their proper sign, and place the result over the given denominator.* See (68) and (70).

Thus, $7\frac{2}{3} = \frac{21}{3} + \frac{2}{3} = \frac{23}{3}$; $a + \frac{m}{n} = \frac{an}{n} + \frac{m}{n} = \frac{an + m}{n}$,

and $x - \frac{x^2 - y^2}{x} = \frac{x^2 - x^2 + y^2}{x} = \frac{y^2}{x}$.

Change the following mixed quantities to improper fractions:

12. $9\frac{4}{7}$. $=\frac{67}{7}$

13. $3x + \frac{x}{y}$. $=\frac{3x^2y}{y}$

14. $b + \frac{ab}{a + b}$. $= a^2 + b^2$

15. $a - \frac{x + a}{a - x}$. a^2

16. $5a^2 + \frac{7b^2}{a^2}$.

17. $a - 5 - \frac{b + x - 25}{a + 5}$.

18. $m - x + \frac{b + x}{a - b}$.

19. $m + x - \frac{b - 5m}{14}$.

20. $a + b - \frac{a^2 + b^2}{a + b}$.

21. $8bx + \frac{5m + 7y}{4a}$.

(*d.*) To reduce an improper fraction to an entire or a mixed quantity,

117. *Divide the numerator by the denominator; and to the quotient, annex the remainder, if any, in the form of a fraction, having for its denominator that of the original fraction.*

Thus, $\frac{a}{a} = 1$; $\frac{19}{5} = 3\frac{4}{5}$; $\frac{ab + b}{a} = b + \frac{b}{a}$.

Reduce the following improper fractions to integers or mixed numbers:

22. $\frac{39}{5}$.

23. $\frac{10}{10}$.

24. $\frac{4ab - b^2}{a}$.

25. $\frac{21ab - 1x}{7a}$.

26. $\frac{35x + 1}{5x}$.

27. $\frac{5ab}{b}$

28. $\frac{a^2 - b^2}{a + b}$.

29. $\frac{12a + ab - 17}{3a}$.

30. $\dfrac{a^2+b-8}{a^2}$.

31. $\dfrac{20m^2(x+y)}{15m^3(x+y)}$.

(*e.*) To reduce fractions to their lowest terms,

118. *Divide both terms of the fraction by their greatest common divisor.* See (100).

Thus, to reduce $\dfrac{14}{63}$ to its lowest terms, we find the greatest common divisor to be 7; hence, $\dfrac{14 \div 7}{63 \div 7} = \dfrac{2}{9}$. The value of the fraction is not changed, see (113), and $\dfrac{2}{9} = \dfrac{14}{63}$. So

$$\frac{5ab - 10a^2x + 15a^3y}{20az - 10a^2m + 5ab} = (95,\ (1))\ \frac{5a(b - 2ax + 3a^2y)}{5a(4z - 2am + b)}$$

$$= \frac{b - 2ax + 3a^2y}{4z - 2am + b}.$$ Also, $$\frac{a+b}{a^2+2ab+b^2} = \frac{1}{a+b}.$$

Reduce the following fractions to their lowest terms:

32. $\dfrac{7a}{35}$.

33. $\dfrac{6b}{42b^2c}$.

34. $\dfrac{ab}{a^2x}$.

35. $\dfrac{3ax}{3ax^2y}$.

36. $\dfrac{ax+a^2y}{a^2b+ax}$.

37. $\dfrac{3axy}{9ax+27xy}$.

38. $\dfrac{24a^2-12}{16ay+20ac}$.

39. $\dfrac{13a}{39x}$.

40. $\dfrac{7xz}{12xy}$.

41. $\dfrac{2a^2b-4a^2x}{6a^3+10a^2y}$.

42. $\dfrac{8axy-4ay+16a^2y}{12axy-20ay+24axyz}$.

43. $\dfrac{ay+by-xy}{ay+cy+dy}$.

44. $\dfrac{6ax^2+12abx+6ab^2}{18a^2x^2-18a^2b^2}$.

(*f.*) To reduce two or more fractions to a common denominator,

119. (1.) *Multiply each denominator by all the other denominators; and to preserve the value of each fraction* (113), *multiply its numerator by the same factors.* Or,

(2.) *Find the least common multiple of all the denominators, for the least common denominator. To obtain the proper multiplier for any numerator, divide the common denominator by the denominator of that fraction.*

Thus, to reduce the fractions $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{5}{7}$ to a common denominator, we must employ the first part of the rule, since the denominators are all prime to each other (88) Then, $\frac{1}{3}=\frac{1\times5\times7}{3\times5\times7}=\frac{35}{105}$; $\frac{2}{5}=\frac{2\times3\times7}{5\times3\times7}=\frac{42}{105}$; and $\frac{5}{7}=\frac{5\times3\times5}{7\times3\times5}=\frac{75}{105}$.

In the same way $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n^2}$ are thus reduced: $\frac{a}{b}=\frac{a\times d\times n^2}{b\times d\times n^2}=\frac{adn^2}{bdn^2}$; $\frac{c}{d}=\frac{c\times b\times n^2}{bdn^2}=\frac{bcn^2}{bdn^2}$; and $\frac{m}{n^2}=\frac{m\times b\times d}{bdn^2}=\frac{bdm}{bdn^2}$. We then have $\frac{adn^2}{bdn^2}$, $\frac{bcn^2}{bdn^2}$, and $\frac{bdm}{bdn^2}$, for the fractions reduced to a common denominator.

But if the denominators have common factors, the second part of the rule should be employed. Thus, to reduce $\frac{4}{9}$, $\frac{5}{12}$, $\frac{7}{18}$, $\frac{1}{6}$, and $\frac{3}{4}$ to a common denominator, we find the least common multiple of the denominators by (99) to be 36. Then, $\frac{4}{9}=\frac{4\times4}{9\times4}$; $\frac{5}{12}=\frac{5\times3}{12\times3}$; $\frac{7}{18}=\frac{7\times2}{18\times2}$; $\frac{1}{6}=\frac{1\times6}{6\times6}$; and

$\frac{3}{4}=\frac{3\times 9}{4\times 9}$. So we have $\frac{16}{36}$, $\frac{15}{36}$, $\frac{14}{36}$, $\frac{6}{36}$, and $\frac{27}{36}$ for the least common denominators.

To reduce $\frac{3a}{x^2-y^2}$ and $\frac{5x}{x+y}$ to the least common denominator, as $x^2-y^2=(x+y)(x-y)$, it must be the least common denominator sought; $\frac{5x}{x+y}=\frac{5x(x-y)}{(x+y)(x-y)}$ $=\frac{5x^2-5xy}{x^2-y^2}$. The first fraction remains unchanged.

$\frac{5}{9ax}$, $\frac{a+b}{8x^2}$, and $\frac{3}{a+b}$ are reduced to the least common denominator as follows. By (99) we find the least common multiple to be $72ax^2(a+b)$. Hence,

$$\frac{5}{9ax}=\frac{5\times 8x(a+b)}{9ax\times 8x(a+b)}=\frac{40ax+40bx}{72a^2x^2+72abx^2}.$$

$$\frac{a+b}{8x^2}=\frac{(a+b)\times 9a(a+b)}{8x^2\times 9a(a+b)}=\frac{9a(a+b)^2}{72ax^2(a+b)}=\frac{9a^2+18ab+9b^2}{72a^2x^2+72abx^2}$$

$$\frac{3}{a+b}=\frac{3\times 72ax^2}{(a+b)72ax^2}=\frac{216ax^2}{72a^2x^2+72abx^2}.$$

Reduce the following fractions to their least common denominator:

By the first method.

45. $\frac{3x}{a^2}$ and $\frac{a^2}{5b}$.

46. $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{a}$.

47. $\frac{3x}{a}$, $\frac{2b}{5c}$, and y. See (115).

48. $\frac{a+b}{m-1}$ and $\frac{a-b}{m+1}$.

49. $\frac{5}{6}$, $\frac{a}{b}$, and $\frac{a+c}{d}$.

50. $\frac{a}{x+5}$ and $\frac{3+y}{2d}$.

51. $8x^2$ and $\frac{3y}{z^2}$.

52. $\frac{2a}{a+b}$ and $\frac{5x}{a-b}$.

By the second method.

53. $\frac{5x^3}{7mn^3}$ and $\frac{7a^2}{42m^2n}$.

54. $\frac{x}{5y}$, $\frac{bc}{cd}$, and $\frac{4}{15}$.

55. $\frac{a+b}{15x}$ and $\frac{c}{60x^3}$.

56. $\frac{4}{a+b}$ and $\frac{5y}{a^2-b^2}$.

57. $\frac{m+n}{m-n}$ and $\frac{m}{m^2-2mn+n^2}$.

58. $\frac{a}{3b}$, $\frac{c}{27x^2}$, and $\frac{5}{18bx}$.

59. $\frac{1}{8a^2}$, $\frac{7}{12}$, and $\frac{5b}{9a}$.

60. $\frac{a-b}{a+b}$ and $\frac{a+x}{a^2+2ab+b^2}$.

61. $\frac{8y}{63a^2}$, $\frac{5x}{21b}$, and $\frac{7}{91b^3}$.

62. $\frac{x+y}{x^2-y^2}$ and $\frac{x-y}{x+y}$.

63. $\frac{m}{m+1}$, $\frac{a+x}{m^2-1}$, and $\frac{4c^2}{5a^2x}$.

SECTION III.

ADDITION OF FRACTIONS.

The sum of two or more fractions, as in entire quantities, is *indicated* by simply writing them with the sign $+$. Thus, $\frac{2a}{b}+\frac{m-n}{x-y}+\frac{c}{d}$, indicates the sum of the respective fractions.

Note.—The sign + or — before a fraction, should always be placed on the same line with that which separates the numerator from the denominator. The sign, it must be remembered, affects the *whole* fraction, and not any single term in either numerator or denominator.

The sum of $\frac{2a}{b}+\frac{3a}{b}$ is evidently $\frac{5a}{b}$; so $\frac{7}{12}+\frac{2}{12}+\frac{1}{12}=\frac{10}{12}$, or $\frac{5}{6}$; and $\frac{3a}{m^2}+\frac{a-m}{m^2}$ is $\frac{3a+a-m}{m^2}=\frac{4a-m}{m^2}$.

Suppose it is required to add $\frac{5a}{3b}$ and $\frac{3c}{b^2x}$. The indicated sum is $\frac{5a}{3b}+\frac{3c}{b^2x}$; but as these fractions have not the same denominator, we cannot add the numerators until we have reduced them to a common denominator, which is $3b^2x$. Then $\frac{5a}{3b}=\frac{5abx}{3b^2x}$, and $\frac{3c}{b^2x}=\frac{9c}{3b^2x}$. The sum will be $\frac{5abx+9c}{3b^2x}$.

Hence we have the following rule for addition:

120. *Reduce the fractions to a common denominator; then add the numerators, placing the result over the common denominator, and reduce the similar terms.*

Thus, to add $\frac{3a-b}{x^2+7-z}$, $\frac{c+2b}{x^2+7-z}$, and $\frac{a-c-b}{3(x^2+7-z)}$, we have, $\frac{9a-3b}{3(x^2+7-z)}+\frac{3c+6b}{3(x^2+7-z)}+\frac{a-c-b}{3(x^2+7-z)}$

$$=\frac{9a-3b+3c+6b+a-c-b}{3(x^2+7-z)}=\frac{10a+2b+2c}{3x^2+21-3z}. \quad \textit{Ans.}$$

1. Add $\frac{a+b}{x^2+y^2}$, $\frac{3a+2b}{x^2+y^2}$, and $\frac{4a-6b}{x^2+y^2}$.

2. Add $\frac{x^2-y+z}{2ab^2}$, $\frac{2y-z-3x^2}{2ab^2}$, and $\frac{2x^2-y+z}{2ab^2}$.

3. Add $\frac{a}{x}$, $\frac{b}{4}$, $\frac{c}{x}$, and $\frac{2d}{x^2}$.

d $\frac{2a}{x^2-y^2}$, $\frac{3b}{x^2-y^2}$, $\frac{3a}{x^2-y^2}$, and $\frac{c}{x-y}$.

5. Add $\frac{a^2-7+2c}{m+n}$, $\frac{3b^3+18+c}{m+n}$, and $\frac{a^2-4b^3-10-3c}{m+n}$.

6. Add $\frac{3a^2}{2b}$, $\frac{4a^2}{3d}$, and $\frac{5a}{4bc}$.

7. Add $\frac{4a}{b}$, $\frac{5b}{c}$, and $\frac{6c}{a}$.

8. Add $\frac{a+b}{5}$ and $\frac{a+6}{8}$.

9. Add $6a^2$, $\frac{x+y}{3}$, and $\frac{3x+y}{7}$.

10. Add $6x$, $\frac{7x}{a}$, 5, and $\frac{3x+2}{4}$.

11. Add $\frac{4x}{9a}$, $\frac{7x-1}{2}$, $\frac{3x+5}{8a^2b^2}$, and $2x-y$.

12. Add $\frac{5a}{m^2-n^2}$ and $\frac{x+y}{m+n}$.

13. Add $\frac{1}{x+y}$ and $\frac{1}{x-y}$.

14. Add ab, 15, $\frac{x+y}{14c^2}$, and $\frac{6}{7}$.

15. Add $\frac{a}{4b}$, $\frac{c}{9d}$, $4x$, and $\frac{5}{6}$.

16. Add $\frac{1}{2}x$ and $\frac{3}{4}x$. See (45).

17. Add $\frac{1}{2}x$, $\frac{2}{3}x$, $\frac{3}{4}x$, and $\frac{5}{6}x$.

18. Add $\frac{1}{4}a$, $\frac{2}{3}a$, and $\frac{1}{2}a$.

19. Add $\frac{2}{3}ab$, $\frac{5}{6}ab$, $\frac{1}{4}ab$, and $\frac{5}{9}abc$.

20. Add $\frac{a}{m+n}$ and $\frac{5a}{m-n}$.

SECTION IV.

SUBTRACTION OF FRACTIONS.

10 To indicate the difference of two fractions, place the sign — between them; thus, $\frac{a+b}{m^2x} - \frac{a-b}{m^2x}$.

As these fractions have the same denominators, we may subtract the numerators; thus, $\frac{a+b-a+b}{m^2x}$ see (70) $= \frac{2b}{m^2x}$.

If the denominators are not the same, they must be reduced to a common denominator before uniting the numerators; thus, $\frac{6x-2}{x+1} - \frac{4x+3}{x-1} = \frac{(6x-2)(x-1)}{(x+1)(x-1)} - \frac{(4x+3)(x+1)}{(x-1)(x+1)}$

$$= \frac{6x^2-2x-6x+2}{x^2-1} - \frac{4x^2+3x+4x+3}{x^2-1}$$

$$= \frac{6x^2-2x-6x+2-4x^2-3x-4x-3}{x^2-1} = \text{reducing,}$$

$\frac{2x^2-15x-1}{x^2-1}$. *Ans.*

Hence, we have the following rule for subtracting fractions:

121. *Reduce the fractions to a common denominator, and subtract the numerators one from the other, changing all the signs in the numerator of the fraction to be subtracted.*

1. Subtract $\frac{a}{b}$ from $\frac{c}{d}$; $\frac{a+b}{7}$ from $6a$.

2. Subtract $\frac{a-d}{4}$ from $9a$; $\frac{4am}{x}$ from $\frac{3bc}{5y}$.

3. Subtract $3y + \frac{a-b}{c}$ from $x + \frac{y-7}{4}$.

NOTE.—We may change these mixed quantities to improper fractions by (116), and then subtract; or we may subtract the enti quantities first, and then the fractions; thus, $x-3y+\frac{y-7}{4}-\frac{a-b}{c}$, and then reduce.

4. Subtract $\frac{3a+5}{8}$ from $\frac{7a}{2}$.

5. Subtract $\frac{1}{a+b}$ from $\frac{1}{a-b}$; $\frac{1}{a-b}$ from $\frac{1}{a+b}$.

6. Subtract $\frac{x+y}{a}$ from $x-\frac{3-y}{b}$.

7. Subtract $x-\frac{a-b}{4}$ from $3y$.

8. Subtract $4y^2$ from $5y^2-\frac{x-8xy}{z}$.

9. Subtract $\frac{3x^2-z}{4z}$ from $\frac{8x^2+y}{z}$.

10. Subtract $\frac{a-m}{6}$ from $\frac{6}{a+m}$.

11. Subtract $\frac{z}{20a}$ from $\frac{x}{5a^3b}$.

12. Subtract $\frac{5ax}{21y}$ from $\frac{3x}{7}$.

13. Subtract $\frac{a-b}{x}$ from $a+b$.

14. Subtract $a-\frac{4+a}{3}$ from $x-y$.

15. Subtract $\frac{5x+6}{m^2-1}$ from $\frac{8}{m+1}$.

16. Subtract $a-b$ from $\frac{27x}{a+b}$.

17. Subtract $\frac{a+b}{a-b}$ from $\frac{a^2+b^2}{a^2-2ab+b^2}$.

18. Subtract $\frac{m^2-2x+d^2}{12a^2x^2}$ from $\frac{m^2+2x+d^2}{18a^2x^2}$.

SECTION V.

MULTIPLICATION OF FRACTIONS.

To multiply a fraction by an entire quantity,

122. *Divide the denominator if possible; if not, multiply the numerator by the entire quantity.* See (111).

Thus, $\frac{5a}{12} \times 4 = \frac{5a}{3}$; here, as the unit is divided into $\frac{1}{4}$ as many parts as at first, each part must be 4 times as great. If we had multiplied the *numerator*, thus, $\frac{5a}{12} \times 4 = \frac{20a}{12}$, or $\frac{5a}{3}$ (118), we should have the same result as before.

As $\frac{5a}{12} \times 4$ will manifestly give the same result as $4 \times \frac{5a}{12}$, see (71, example), the preceding rule applies also to the multiplication of an *entire quantity by a fraction.* In the following examples, the result should always be reduced to its lowest terms. See (118).

Thus, $4x \times \frac{3a}{2bcx} = \frac{3a \times 4x}{2bcx} = \frac{12ax}{2bcx} = \frac{6a}{bc}$.

Here we might have reduced before multiplying by cancelling the common factors $2x$; thus, $\frac{3a \times 2}{bc} = \frac{6a}{bc}$.

1. Multiply $\frac{7x}{12a}$ by $5y$.

2. Multiply $\frac{a^2}{m}$ by bc^2d.

3. Multiply $\frac{a+b}{7x}$ by $4xy$.

4. Multiply $\frac{a-2}{b+c}$ by $x-2$.

5. Multiply $\frac{4ab-6}{5x}$ by $5xy^2$.

6. Multiply $-5ay$ by $\frac{8ax-4b}{5y}$.

Note.—Cancelling like factors, we shall find that the numerator is to be multiplied by $-a$.

7. Multiply $2b$ by $\frac{ad}{2bx}$; 4 by $\frac{3m}{8}$.

8. Multiply $\frac{2x-6b^2x}{7z}$ by $-14az$.

9. Multiply 7 by $\frac{3a-7b}{21x+14y}$.

10. Multiply $5b$ by $\frac{8+a}{15b-10bx}$.

11. Multiply $\frac{a-x}{xy-2x}$ by $-x$.

12. Multiply $\frac{a+b}{x^2-y^2}$ by $x+y$; by $a+b$.

13. Multiply $\frac{6x-4m}{15a^2x-9am+18a^3b}$ by $3abx$.

14. Multiply $\frac{a}{5}$ by 5. See ([illegible]).

15. Multiply $\frac{7}{a+b}$ by $a+b$.

16. Multiply $\frac{8x}{15b^2c^3-5ab^2c^3}$ by $5b^2c^3(3-a)$.

17. Multiply $\frac{5y}{a^2+2ab+b^2}$ by $(a+b)^2$.

18. Multiply $\frac{m^2-2mn+n^2}{(a+b)(a-b)}$ by a^2-b^2.

To multiply a fraction by a fraction, is the same as to take a fraction of a fraction; as $\frac{2}{3}$ of $\frac{3}{4}$, $\frac{4}{5}$ of $\frac{5}{8}$, is the same as $\frac{2}{3} \times \frac{3}{4}$, $\frac{4}{5} \times \frac{5}{8}$.

123. *A fraction of a fraction is called a* COMPOUND FRACTION.

What is $\frac{2}{3}$ of $\frac{4}{5}$?

To find one-third of $\frac{4}{5}$, we must divide the fraction by 3; that is, we multiply the denominator by 3 (112); then, since $\frac{1}{3}$ of $\frac{4}{5} = \frac{4}{15}$, two-thirds must be twice as much, or $\frac{8}{15}$. Here we have multiplied the numerator (111) by 2. Then $\frac{2}{3}$ of $\frac{4}{5}$, or $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$; that is, we multiply the corresponding terms of the fractions together. Hence, to multiply a fraction by a fraction,

124. *Multiply the numerators together for the numerator of the product; and the denominators together for the denominator of the product. Reduce the result to its lowest terms.*

REMARK.—The common factors may be omitted before multiplying: thus, $\frac{14bx}{15y^2} \times \frac{9dm}{4bx} \times \frac{5ay}{7d^2m^2} = \frac{3a}{2dmy}$.

19. Multiply $\frac{a+x}{y^2}$ by $\frac{x}{b^2-4}$. $= \frac{ax + x^2}{b^2y^2 - 4y^2}$

20. Multiply $\frac{2c}{ab}$ by $\frac{3z^2}{4}$. $= \frac{3cz^2}{2ab}$

21. Multiply $\frac{a-b}{y}$ by $\frac{3y}{2x}$. $\frac{3a - 3b}{2x}$

22. Multiply $\frac{a+b}{3+z}$ by $\frac{y-1}{a+b}$. $\frac{y-1}{3+z}$

23. Multiply $\frac{a^2-x^2}{4}$ by $\frac{2bc}{a-x}$. $\frac{abc+bcx}{2}$

24. Multiply $\frac{axy}{b-8}$ by $\frac{b-8}{axy}$. 1

25. Multiply $\frac{3a}{8}$ of $\frac{m+n}{5(x+y)}$ by $\frac{10m}{6a}$ of $\frac{4x}{m+n}$. $\frac{mx}{2x+2y}$

NOTE.—Indicate the multiplication before reducing thus: $\frac{3a}{8} \times \frac{(m+n)}{5(x+y)} \times \frac{10m}{6a} \times \frac{4x}{m+n}$.

$-3my + xy$

26. Multiply $\frac{4}{5}$ of $\frac{3am-ax}{2m}$ by $\frac{10}{8}$ of $\frac{x-y}{3a}$. $\frac{3mx - x^2}{6m}$

27. Multiply together $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{x}{y}$. $\frac{ax}{cy}$

28. Multiply $\frac{12a^2b-6ax}{m^2-n^2}$ by $\frac{m+n}{18am-12a}$.

29. What is the product of $\frac{7x^2}{36y^3}$ of $\frac{9am}{14z}$, and $\frac{2y^2z}{5mn}$.

30. Multiply $\frac{5x^2-10xy+5y^2}{m+n}$ by $\frac{m^2-n^2}{15x-15y}$.

31. Multiply $3\frac{4}{5}$ by $a+\frac{x}{y}$.

NOTE.—Reduce the mixed numbers to a fraction before multiplying. See (116).

32. Multiply $x+\frac{x-y}{m}$ by $\frac{m^2}{x^2-y^2}$.

33. Multiply $5am+\frac{d-a^2m}{3a}$ by $x+\frac{4a^2}{3}$.

SECTION VI.

DIVISION OF FRACTIONS.

(*a*) To divide a fraction by an entire quantity,

125. *Divide the numerator by the entire quantity if possible; if not, multiply the denominator.* See (112).

Thus, $\frac{5x}{12} \div 5 = \frac{x}{12}$; $\frac{bc}{y} \div bc = \frac{1}{y}$; but $\frac{5x}{12} \div 2 = \frac{5x}{24}$; and $\frac{bc}{y} \div a = \frac{bc}{ay}$.

1. Divide $\frac{3ab}{x}$ by $3a$.

2. Divide $\frac{x+y}{bc}$ by $x+y$.

3. Divide $\frac{8xy}{b-c}$ by $4y$.

4. Divide $\frac{14a^2}{3xy}$ by $7a$.

5. Divide $\frac{16a^2 - 8a}{a^2 + 2x}$ by $4a$.

6. Divide $\frac{3x^2y}{a-c}$ by $3x$.

7. Divide $\frac{32x - 8bx}{a^2 + 8}$ by $8m$. $\frac{4x - bx}{a^2m + 8m}$

8. Divide $\frac{7a^2x}{y-z}$ by $3b^2$.

9. Divide $\frac{2m^2 - mx}{z}$ by $3m^2z$.

10. Divide $\frac{8b - 32xy}{x+8}$ by 4.

(*b*) To divide an entire quantity by a fraction.

In this case, we can change the entire quantity to a fraction of the same denomination with the given fraction (115). Thus, $6 \div \frac{3}{4} = \frac{24}{4} \div \frac{3}{4}$, which is the same as $24 \div 3$, or $\frac{24}{3} = 8$. Now if we had multiplied 6 by $\frac{4}{3}$, (or the divisor inverted), we should have had the same result, $\frac{24}{3}$. So $m \div \frac{a}{b} = \frac{mb}{b} \div \frac{a}{b} = \frac{mb}{a}$, and $m \times \frac{b}{a}$ (or $\frac{a}{b}$ inverted) $= \frac{mb}{a}$, as before.

(*c*) To divide a fraction by a fraction.

We can change both fractions to a common denominator (119), and then divide. Thus, $\frac{3}{5} \div \frac{2}{7} = \frac{21}{35} \div \frac{10}{35} = \frac{21}{10}$. But, by inverting the divisor, and multiplying, we have $\frac{3}{5} \times \frac{7}{2}$, also equal to $\frac{21}{10}$. So $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bc}$; and $\frac{a}{b} \times \frac{d}{c}$ (or $\frac{c}{d}$ inverted) also equals $\frac{ad}{bc}$.

Hence, to divide by a fraction,

126. *Invert the divisor, and proceed as in multiplication.* See (124).

REMARK.—Entire and mixed quantities must always be reduced to fractions before applying this rule. Thus, $m \div \frac{a}{b} = \frac{m}{1} \times \frac{b}{a} = \frac{bm}{a}$; $\frac{a+b}{5x} \div a^2 - b^2 = \frac{a+b}{5x} \times \frac{1}{a^2 - b^2} = \frac{a+b}{5x(a^2 - b^2)} = \frac{1}{5x(a-b)}$; and $\left(m + \frac{a}{x}\right) \div \left(3 + \frac{b}{c}\right) = \frac{mx + a}{x} \div \frac{3c + b}{c} = \frac{mx + a}{x} \times \frac{c}{3c + b} = \frac{cmx + ac}{3cx + bx}$.

Divide $\frac{2a^2x^3}{m^2 - y^2}$ by $\frac{14ax}{m + y}$. Inverting the divisor and multiplying, we have $\frac{2a^2x^3}{m^2 - y^2} \times \frac{m + y}{14ax} = \frac{ax^2}{(m - y) \times 7} = \frac{ax^2}{7m - 7y}$.

11. Divide $\frac{2a}{3x}$ by $\frac{b}{3y}$. =

12. Divide $\frac{3a - b}{\cancel{c}}$ by $\frac{b}{\cancel{c}(x - y)}$.

13. Divide $\frac{8(a^2 - b^2)}{\cancel{xy}}$ by $\frac{2(a + b)}{\cancel{xy}}$.

14. Divide $\frac{4\cancel{a}b}{x(a + 1)^5}$ by $\frac{az}{8(\cancel{a} + 1)^3}$.

15. Divide $\frac{a + b}{x + y}$ by $\frac{\cancel{a + b}}{y - 4}$.

16. Divide $a - \frac{ax}{x + y}$ by $b + \frac{3bx}{x + y}$.

17. Divide $\frac{3}{4}$ of $\frac{5a}{9}$ by $\frac{a}{5}$ of $\frac{m}{n}$.

18. Divide $\cancel{xy}$ by $\frac{\cancel{4xy} - 16x}{8a - 12}$.

19. Divide $\frac{-x}{y}$ by $\frac{a}{-b}$.

20. Divide $9m^2 - 27m^3$ by $\frac{18am}{5z}$.

21. Divide $\frac{5(a + b)}{7a^2}$ by $10ax$.

22. Divide $\frac{11(x^2 - y^2)}{8(m + n)}$ by $\frac{55(x + y)}{24(m^2 + 2mn + n^2)}$.

If we should indicate by a fraction the quotient in any of the previous examples, it would give rise to a COMPLEX FRACTION, that is, to a fraction having another fraction in one or in both of its terms. Thus, $\frac{\frac{2a}{3x}}{\frac{b}{3y}}$, $\frac{\frac{a}{b}}{m}$, $\frac{x}{\frac{5a}{4}}$, and $\frac{3\frac{1}{2}}{4\frac{1}{3}}$, are all complex fractions. To reduce such fractions, consider the numerator as a dividend, and the denominator as a divisor (109), and apply the preceding rule. Thus, $\frac{3\frac{1}{2}}{4\frac{1}{3}}=\frac{7}{2}\div\frac{13}{3}=\frac{7}{2}\times\frac{3}{13}=\frac{21}{26}$. So $\frac{\frac{x}{y}}{n}=\frac{x}{y}\div\frac{n}{1}=\frac{x}{y}\times\frac{1}{n}=\frac{x}{ny}$.

Reduce the following complex fractions:

23. $\frac{7x}{\frac{2}{3}}$.

24. $\frac{\frac{5a}{2}}{6}$.

25. $\frac{3\frac{7}{8}}{5\frac{1}{4}}$.

26. $\frac{\frac{6x}{m}}{a+b}$.

27. $\frac{x+\frac{x}{2}}{b-\frac{m}{n}}$.

28. $\frac{\frac{7bc}{5n}}{\frac{14ab}{15m^2n}}$.

127. *The* RATIO *which one quantity bears to another is often expressed by a fraction.*

Thus, if two quantities are to each other as 1 to 2, or in the *ratio* of 1 to 2, the smaller is $\frac{1}{2}$ the larger; the ratio of 2 to 3 is expressed by the fraction $\frac{2}{3}$; the ratio of m to n by $\frac{m}{n}$.

29. What is the ratio of $2x$ to $3x$? of $a+b$ to $a-b$?

30. What is the ratio of $15a^2$ to $18x$? of $7xy$ to $28y^2$?

31. What is the ratio of $\frac{13x^2y^3}{5(a+b)}$ to $\frac{91xy^3}{8x^2(m-n)}$?

32. What is the ratio of $\frac{16(m^2-n^2)}{xy}$ to $8x^2(m-n)$?

CHAPTER IV.

EQUATIONS OF THE FIRST DEGREE.

SECTION I.

INTRODUCTION.

128. *An* EQUATION *is an expression* (6) *for the equality of two algebraic quantities.*

Thus, $8 + 4 = 18 - 6$, and $x + 5 = 2x - 27$, are equations.

129. *An equation of the* FIRST DEGREE *contains only the first power of the unknown quantity* (19).

Thus, $x + 3 = y - 2$ is an equation of the first degree.

130. *An equation of the* SECOND *or the* THIRD DEGREE *contains the second or the third power of the unknown quantity.*

Thus, $x^2 + x = 12$ is an equation of the second, and $x^3 = y^2 + 18$ is an equation of the third degree.

131. *An equation consists of two parts, called its* MEMBERS (17); *that on the left of the sign = is called the first or left hand member; that on the right, the second or right hand member.*

132. *When the two members are increased or diminished equally, their* VALUE *is changed, but their* EQUALITY *is still preserved.*

We may, then, without destroying the equality,

(1.) ADD *the same or equal quantities to both members* (24);

(2.) SUBTRACT *the same or equal quantities from both members* (25);

(3.) MULTIPLY *both members by the same or equal quantities* (26);

(4.) **Divide** *both members by the same or equal quantities* (27).

Remark.—The learner should bear in mind that, in all operations upon equations, *each entire member* is to be regarded as a single quantity. Hence, to multiply or divide a member, we must multiply or divide *each of its terms* (81), (89). In general, the members should be reduced, if possible (66).

133. *An equation is said to be* SOLVED *when the unknown quantity is found in terms of known quantities.*

134. *To solve an equation, we must*

(1.) *Bring all the terms containing the unknown quantity together into one member;*—

(2.) *Free them from all connection with known quantities.*

135. *A known quantity may be connected with an unknown quantity,*—

(1.) *By* ADDITION; *as,* $x + 25 = 50$.

(2.) *By* SUBTRACTION; *as,* $x - 12 = 36$.

(3.) *By* MULTIPLICATION; *as,* $5x = 30$.

(4.) *By* DIVISION; *as,* $\frac{x}{15} = 4$.

(5.) *By any or all of these combined; as,* $3x + 25 - \frac{x}{3} - 27 = 6$.

136. *To free the unknown quantity from the known,* REVERSE *the operation by which it is combined.*

(1.) Thus, in $x + 25 = 50$, *subtract* (132) 25 from both members.

(2.) In $x - 12 = 36$, *add* (132) 12 to each member.

(3.) In $5x = 30$, *divide* both members (132) by 5.

(4.) In $\frac{x}{15} = 4$, *multiply* both members (132) by 15.

Performing these several operations, we shall have,

(1.) $x + 25 - 25 = 50 - 25$, or $x = 25$.

(2.) $x - 12 + 12 = 36 + 12$, or $x = 48$.
(3.) $5x \div 5 = 30 \div 5$, or $x = 6$.
(4.) $\frac{x}{15} \times 15 = 4 \times 15$, or $x = 60$.

These various operations are sometimes combined, as in the equation given above

$$3x + 25 - \frac{x}{3} - 27 = 6. \quad \text{Reducing (66),}$$
$$\frac{8x}{3} - 2 = 6. \quad \text{Multiplying by 3,}$$
$$8x - 6 = 18. \quad \text{Adding 6 to each member,}$$
$$8x = 24. \quad \text{Dividing by 8,}$$
$$x = 3.$$

Hence, to solve any given equation of the first degree consisting of entire quantities only,

137. *Transpose all the terms containing the* UNKNOWN *quantity to the* FIRST *member, and all the terms containing* KNOWN *quantities to the* SECOND, *changing the sign of every term transposed* (33).

138. *Unite all the terms in the first member, also all the terms in the second* (66)*; and divide both members by the coefficient of the unknown quantity.*

If the equation contains fractions,

139. *Clear the equation of fractions by* MULTIPLYING *both members by the least common multiple* (99) *of the denominators of the fractions; and then transpose and reduce as above.*

140. *At all stages, the members should be* REDUCED (66) *whenever it is possible. Terms consisting of abstract numbers can always be united; and thus the operations on the equation simplified.*

141. *The two members of the equati[illegible] must always be of the* SAME KIND; *that is, we must have dollars equal* DOLLARS; *shillings* = SHILLINGS; *hours* = HOURS; *miles* = MILES, &c.

Every equation may be considered as an algebraic statement of some problem or question. No definite rule can be given for putting a problem or question into an equation, but a consideration of the following example may aid the learner:

A gentleman gave $175 for his watch, chain, and seal; the chain cost him twice as much as the seal, and the watch twice as much as the chain. What was the price of each?

Now here are three unknown quantities, the price of the watch, the price of the chain, and the price of the seal. We will let x represent each of these quantities in turn, and solve the equations arising from each of the statements of the question.

Let $x =$ the price of the *seal*,
then $2x =$ the price of the chain,
and $4x =$ the price of the watch.
$x + 2x + 4x = \$175$, by the conditions of the question.
(reducing) $7x = 175$. Dividing by 7,
$x = \$25$, the price of the seal.
$2x = \$50$, the price of the chain.
$4x = \$100$, the price of the watch.

Proof. $100 + 50 + 25 = 175$, as above

Now let $x =$ the price of the *chain*,
then $\frac{x}{2} =$ the price of the seal,
and $2x =$ the price of the watch.

Then $x + \frac{x}{2} + 2x = 175$. Clearing of fractions (139),
$2x + x + 4x = 350$. Reducing,
$7x = 350$. Dividing by 7,
$x = 50$, the price of the chain.
$\frac{x}{2} =$ [illegible] the price of the seal.
$2x = 100$, the price of the watch.

Again, let $x =$ the price of the *watch*,

then $\frac{x}{2} =$ the price of the chain,

and $\frac{x}{4} =$ the price of the seal.

$x + \frac{x}{2} + \frac{x}{4} = 175.$ Clearing of fractions,

$4x + 2x + x = 700.$ Reducing,

$7x = 700.$ Dividing,

$x = 100,$ the price of the watch,

$\frac{x}{2} = 50,$ the price of the chain,

$\frac{x}{4} = 25,$ the price of the seal.

We need to proceed in all examples as in this. That is, we must first obtain a clear idea of the *meaning* of the question proposed, in order to determine *what* is the thing required or the *unknown quantity*. Having done this, let x represent that quantity, and *put the question into an equation*, by performing all the operations upon x, which the conditions of the question may require. The value of x is then found by solving this equation according to the principles already given.

SECTION II.

EQUATIONS OF THE FIRST DEGREE CONTAINING ONE UNKNOWN QUANTITY.

1. If a certain number be multiplied by 5 and by 8, the sum of the products will be 91. What is that number?

Let $x =$ the number,

then $5x + 8x = 91.$ Reducing,

$13x = 91.$ Dividing by 13,

$x = 7.$ *Ans.*

It is evident that we may make any number of similar questions by varying the multipliers and the sum of the products. Thus, if the number be multiplied by 7 and by 9, and the sum of the products be 96, we shall have the equation $7x + 9x = 96$, and $x = 6$. It is well to embrace all these various questions in one, by employing symbols for any numbers (51) that may be used. The question will then be stated thus:

2. If a certain number be multiplied by m and by n, the sum of the products will be a. What is that number?

$$\text{Let } x = \text{the number,}$$

$$\text{then } mx + nx = a.$$

The first member, $mx + nx$, may be separated into two factors (93, (1)), one of which is x and the other $m + n$, which is the coefficient of x. Then,

$$(m + n)x = a. \quad \text{Dividing by } m + n \text{ (138),}$$

$$x = \frac{a}{m + n}.$$

This is the formula (53) for the value of x; interpreting which in words (5, *Note*), we obtain the following rule for all similar cases:

142. *Divide the sum of the products by the sum of the multipliers; the quotient will be the number required.*

Thus, to find a number, such that if multiplied by 5 and by 12 the sum of the products will be 170, we may either use the rule, or substitute these numbers (54) in the formula $\frac{a}{m + n}$. Substituting, we shall have $\frac{170}{5 + 12} = \frac{170}{17} = 10$, the number sought.

As this formula furnishes a *general* rule for all similar examples, the question is said to be *generalized.*

In generalization, it will be seen that *letters* only are used, and that they represent both known and unknown quantities. Such operations belong to *pure* Algebra, and the results will always be general formulas for all similar examples, and, when interpreted, form *rules* for their solution. It will be seen that any algebraic problem may be generalized, though some contain principles of more general application than others. Hence,

143. GENERALIZATION *is an algebraic process, by which we derive a general formula for any given problem.*

NOTE.—When the learner is required to generalize any problem, he should also interpret (4) each symbol, each combination and the result, thereby making a rule; and then illustrate its application by some particular example.

3. Two men, A and B, trade in company, and gain $141, of which B is to have twice as much as A. What is the share of each?

4. Generalize this question by letting a represent the gain, and supposing B to have n times as much as A.

Then if $x =$ A's share,

$nx =$ B's share,

and $x + nx = a$. Separating into factors,

$(1 + n)x = a$. Dividing by $1 + n$,

$$x = \frac{a}{1 + n} = \text{A's share.}$$

$$= \frac{a}{1 + n} \times n = \frac{an}{1 + n} = \text{B's share.}$$

From these formulas, we derive the following rule for all cases where one partner is to have a certain number of times as much as the other.

144. *Add* 1 *to the number which expresses the proportion of one partner's share to that of the other, and divide the whole gain by the sum for the share of the first. This, multiplied by the number which expresses the proportion of the second, will be the share of the second.*

5. Divide $2800 between A and B, giving B 6 times as much as A. $x =$ [illegible]

Note.—Substitute in the preceding formulas.

6. Divide n into four parts, of which the second shall be a, the third b, and the fourth c times the first.

7. A man distributed 70 cents among four poor persons; giving the second twice, the third three times, and the fourth four times, as much as he gave the first. What did he give to each? $x = 7$ — $2x = 14$. [illegible]

Note.—Substitute as above.

8. Said a father to his son, "Our joint ages are 78 years, and I am 5 times as old as you." What were their ages? $= 13$ $= 65$

9. Three men, A, B, and C, trade in company, and gain $696, of which B is to receive 3 times as much as A, and C as much as both A and B. What is the share of each? [illegible]

10. Generalize this question by letting a represent the gain. Let B receive m times as much as A, and C receive as much as A and B, and find a formula for the share of each.

11. A farmer hired two men and a boy to do a certain piece of work, agreeing to pay one of the men 5 shillings, the other 4 shillings, and the boy 3 shillings a day. When the work was finished, he paid them $54. How many days were they employed? See (141). $x = 27$

12. A farmer sold an equal number of oxen, cows, and sheep for $632. For the oxen he received $47 apiece; for the cows, $25; and for the sheep, $7. How many did he sell of each sort?

13. A merchant, failing in trade, owes to A, B, C, and D $3597. To B he owes twice as much as to A; to C, as much as to A and B; and to D, as much as to B and C. How much does he owe to each of them? $x =$ [illegible]

14. A boy bought 2 oranges, 3 pears, and 4 apples for 22

cents. He gave as much for a pear as for 2 apples; and twice as much for an orange as for a pear and an apple. What was the price of each?

15. The age of A is double that of B; the age of B is three times that of C; and the sum of all their ages is 140 years. What is the age of each?

16. Generalize by letting A's age be m times B's, and B's n times C's; and the sum of their ages be a years.

17. A man left an estate of \$60,000, to be so divided between his widow, 3 sons, 2 daughters, and a ward, that each daughter should receive twice as much as the ward, each son as much as the ward and a daughter, and the widow twice as much as each son. What was the share of each?

18. How long will it take two men to build 387 rods of wall, if one build 4 and the other 5 rods a day?

Let $x =$ the number of days.

19. Generalize the preceding example, by letting a represent the number of rods in the wall, and m and n the number each could build respectively a day.

20. Four brothers gained, in a year, \$4755; of which B gained three times as much as A; C gained as much as A and B; and D gained as much as B and C. What sum was gained by each?

21. In how many hours will a cistern, containing 264 gallons, be emptied by 3 cocks; one of which discharges 2 gallons in 15 minutes; the second, 5 gallons in 30 minutes; and the third, 3 gallons in 45 minutes?

Note.—Find how many gallons each discharges in an *hour*.

22. One man leaves New York for Boston, and travels 9 miles an hour; another man at the same time leaves Boston for New York, and travels 7 miles an hour. In how many hours will they meet, the cities being 224 miles apart?

23. Four boys, A, B, C, and D, upon counting their money, found they all had $30; of which sum A's share was three times greater than B's; C's share was equal to B's, and one-third of A's; and D's share was equal to A's, and half of C's. What was the share of each?

10x = 30

24. A asked B how much money he had; B replied, that if he had seven times as much, he could lend four times what he then had, and have $69 left. How much had he? x = 23

25. Generalize the last example by supposing that if B had a times as much, he could lend b times what he had, and have c dollars left.

26. A father is seven, and a mother five times as old as their son; and the difference of their ages is 16 years. How old is the son?

27. A man directed, in his will, that his property should be so divided that his son should have three times as much as his daughter, and his widow twice as much as both her children; by which division she received $8235 more than the son. What was the share of each?

28. A farmer employed two men to build 105 rods of wall; one of whom could build 4 rods a day, and the other 3. How many days did they work?

29. A laborer, who spent every week as much as he earned in 2 days, saved 32 dollars in 4 weeks. What were his daily wages?

30. A man left an estate of $21,546; one-third of which he bequeathed to his widow, and directed the remainder to be so divided between his 2 sons and 2 daughters, that each son might receive as much as both the daughters. What was the portion of each?

31. Three travellers found a purse containing $54; of which B secured three times as much as A, and C secured half as much as both of the others. What was the share of each?

32. Two men, A and B, start from the same place, and travel the same way; A at the rate of 45 miles a day, and B at the rate of 30. In how many days will they be 300 miles apart? 20

33. If they were to travel in different directions, in how many days would they be 300 miles apart? 4

34. Generalize both these last questions, by letting *a* and *b* represent the respective rates a day, and *d* represent the distance they were to be apart.

Problems in Interest.

We will conclude this section by generalizing the problems in simple interest.

35. Find the interest of $20 for 5 years at 6 per cent.

The principal, $20, multiplied by .06, the rate for one year, would be the required interest for one year. If it were required to find the interest for 5 years, then $\$20 \times .06 \times 5$ would represent the interest.

To generalize this question,

Let p represent the principal,
t the time,
r the rate, and
i the interest.

Then as $r \times p$ equals the interest for one year,
$t \times r \times p$ will equal the interest for t years. Hence,
$i = trp$.

This formula translated into words, gives the following rule:

145. *The* INTEREST *is equal to the principal, multiplied by the time and rate.*

From the equation $i = trp$, or $trp = i$, we may find successively the values of t, r, and p. Thus,

$$t = \frac{i}{rp}; \text{ that is,}$$

a = p + trp

a = (1

146. *The* TIME *is equal to the interest divided by the product of the rate and principal.*

36. The interest, time, and rate being given, find a formula for the principal. $p = \frac{i}{tr}$

37. The interest, time, and principal being given, find a formula for the rate. $r = \frac{i}{tp}$

38. Required the interest of $350.16 for 4 years, at 5 per cent.

Note.—Substitute in the proper formula. 70.03

39. The interest being $148.10, the rate 5 per cent., and the principal $740.50, what is the time? 4 years

40. Required the principal, the interest being $26.25, the rate 6 per cent., and the time $3\frac{1}{2}$ years. 125

41. Required the rate, the principal being $286.25, the interest $34.35, and the time 2 years. 6 [illegible]

42. What is the amount of $756.26 for $4\frac{1}{4}$ years, at $5\frac{1}{2}$ per cent.? [illegible]

To generalize this question, let a represent the amount; then since the amount is found by adding the principal to the interest,

$$a = p + trp. \quad \text{Separating into factors,}$$
$$a = p(1 + tr); \text{ that is,}$$

147. *To find the* AMOUNT, *add* 1 *to the product of the time and rate, and multiply the sum by the principal.*

Perform the example given above by this rule.

43. What principal will amount to $517.73 in 3 years, at 5 per cent.? [illegible]

Note.—The principal, in this case, is called the *present worth* of $517.73 due in 3 years at 5 per cent. To find a formula for the present worth, we change the equation $a = p + trp$, member for member, thus, $p + trp = a$; or,

$$p(1 + tr) = a. \quad \text{Dividing by } 1 + tr,$$

$$p = \frac{a}{1 + tr}. \quad \text{Hence,}$$

148. *To find the* PRESENT WORTH, *aad* 1 *to the product of the time and rate, and divide the amount by the sum.*

Perform example 43 by the above formula.

44. The amount being \$500, the principal \$400, and the rate 5 per cent., what is the time? 5 years

In the equation $p + trp = a$, transpose p.

$$trp = a - p. \quad \text{Divide by}$$

$$t = \frac{a - p}{rp}$$

Note.—Substitute in the formula; interpret all the symbols (5. Note).

This formula for t is evidently the same with $\frac{i}{rp}$, since the principal, subtracted from the amount, gives the interest.

45. The amount being \$700.50, the principal \$350.25, and the time 25 years, what is the rate?

Note.—Make a formula for the rate and substitute as above.

46. In what time will \$100 be doubled at 6 per cent.?

Here $i = p$, hence $p = trp$, or $1 = tr$, $t = \frac{1}{r}$.

SECTION III.

EQUATIONS OF THE FIRST DEGREE.

1. What number is that, which, being increased by $\frac{1}{2}$, $\frac{3}{4}$, $\frac{2}{7}$, and $\frac{1}{14}$ of itself, becomes 146?

Let $x =$ the number.

Then $x + \frac{x}{2} + \frac{3x}{4} + \frac{2x}{7} + \frac{x}{14} = 146$. Multiplying by 28 (139), we have $28x + 14x + 21x + 8x + 2x = 4088$. Reducing,

$$73x = 4088.$$

$$x = 56.$$

28 is the least common multiple of the denominators, 2, 4, 7, 14. We might have indicated the product in the second member thus: 146×28. Then dividing by 73 we should have $2 \times 28 = 56$ for the value of x. There are various similar means of shortening labor in the solution of equations, which the learner will acquire as he advances.

2. What number is that whose mth part exceeds its nth part by a?

Let $x =$ the number.

$$\text{then } \frac{x}{m} - \frac{x}{n} = a. \qquad \text{Clearing of fractions,}$$

$$nx - mx = amn. \qquad \text{Separating into factors.}$$

$$x(n - m) = amn. \qquad \text{Dividing by } n - m \text{ (138),}$$

$$x = \frac{amn}{n - m}.$$

3. What number is that, whose sixth part exceeds its eighth part by 20?

Note.—Substitute in the above formula.

4. Says A to B, "If to my age $\frac{1}{2}$ and $\frac{3}{4}$ of my age be added, the sum will be 81 years; what is my age?"

5. Generalize this example, thus: if to my age $\frac{1}{b}$ and $\frac{a}{d}$ of my age be added, the sum will be c years. What is my age?

6. A man, having spent three-fifths of his estate, had $978 left. How much had he at first?

7. Generalize the 6th, by letting $\frac{m}{n}$ represent the part of the estate he spent, and a represent the part he had left.

8. A man spent $\frac{1}{3}$ of his life in England, $\frac{1}{4}$ of it in Scotland, and the remainder of it, which was 20 years, in the United States. What was his age?

9. What number is that, $\frac{1}{3}$ of which is greater than $\frac{2}{7}$ of it by 21?

10. Generalize thus; $\frac{1}{m}$ is greater than $\frac{p}{q}$ of it by a. $\frac{amq}{q-mp}$

11. A man, driving his geese to market, was met by another, who said, "Good morrow, master, with your hundred geese." Said he, "I have not a hundred; but if I had as many more, and half as many more, and two geese and a half, I should have a hundred." How many had he? 39

12. If $\frac{2}{7}$ of a ship cost $4000, what is the whole ship worth?

13. Generalize by letting the $\frac{m}{n}$ cost b dollars. (5. Note.) $\frac{bn}{m}$

14. A man sold 75 bushels of wheat to two persons; to one, $\frac{1}{4}$, and to the other, $\frac{3}{8}$ of all he had. How many bushels had he?

15. A man gave to three poor persons $6; to the first, $\frac{1}{3}$, to the second, $\frac{1}{4}$, and to the third, $\frac{1}{6}$ of all the money he had in his pocket. How much had he? 8

16. A and B divide $320 between them, of which B has three times and $\frac{4}{7}$ as much as A. How much has each? See (116).

17. A says to B, "Your age is twice and $\frac{3}{8}$ of my age, and the sum of our ages is 54 years." What is the age of each?

18. If you divide $50 between two persons, giving one $\frac{3}{7}$ as much as the other, what will be the share of each?

15

19. A stranger in Boston spent, the first day, $\frac{1}{3}$ of the money he brought with him; the second day $\frac{1}{4}$; and the third day, $\frac{1}{5}$; when he had only $26 left. How much money did he bring?

20. A man, having invested $\frac{3}{8}$ of his property in bank stock, by which he lost $\frac{2}{3}$ of the sum invested, had stock worth $723 remaining. How much property had he at first?

$$x = \text{sum he had at first.}$$

$$\frac{3x}{8} = \text{invested in bank stock.}$$

$$\tfrac{2}{3} \text{ of } \frac{3x}{8} = \frac{x}{4} = \text{loss.}$$

$$\frac{3x}{8} - \frac{x}{4} = 723.$$

21. A merchant retired from business when he had passed $\frac{9}{10}$ of his life, during $\frac{2}{3}$ of which period he had been engaged in trade, having commenced at the age of 21. At what age did he die?

22. The age of A is $\frac{1}{2}$ that of B, and the age of C is $\frac{1}{3}$ that of A, and the sum of all their ages is 120 years. What is the age of each?

23. Three boys spent 98 cents for fruit. B spent $\frac{5}{8}$ as much as A, and C spent $\frac{1}{5}$ as much as B. What did each spend?

24. If you divide \$75 between two men, in the ratio of 2 to 3, what will each man receive?

NOTE.—The ratio of 2 to 3 is expressed by the fraction $\frac{2}{3}$, see (127); but in stating the question, we must be careful to notice whether x represents the larger or the smaller of the two quantities.—If, in this example, x denote the *larger* sum, then the smaller will be $\frac{2}{3}$ of x or $\frac{2x}{3}$; but if x denote the *smaller*, then the larger will be $\frac{3}{2}$ of x or $\frac{3x}{2}$. Or, in general, if two quantities are to each other as a to b, b being larger than a, than the larger will be the $\frac{b}{a}$ part of the smaller, and the smaller the $\frac{a}{b}$ part of the larger.

25. Generalize the above example, by having the smaller part to the larger, as m to n; and the sum of the two parts equal to a.

26. Divide 84 into two numbers, which shall be to each other as 7 to 5. 49 a 35

27. What two numbers are to each other as 4 to 9, their sum being 91? 28 63

28. Three men trade in company, and gain $780. A put in $2 as often as B put in $3 and C put in $5. What part of the gain must each one receive? 156. B 234. C 390

NOTE.—As A put in $2 as often as B put in $3 and C put in $5; their shares of the stock must be respectively as 2, 3, and 5. Hence the *gain* must be divided in the same ratio; that is, if x represents A's share, $\frac{3x}{2}$ will represent B's, and $\frac{5x}{2}$ C's.

29. If A puts in a dollars, as often as B does b dollars and as C does c dollars, how shall the whole gain, which is n dollars, be divided?

If $x =$ A's share of the gain,

$$\text{then } x \times \frac{b}{a} \text{ or } \frac{bx}{a} = \text{B's},$$

$$\text{and } x \times \frac{c}{a} \text{ or } \frac{cx}{a} = \text{C's},$$

$$x + \frac{bx}{a} + \frac{cx}{a} = n. \quad \text{Multiply by } a,$$

$$ax + bx + cx = an. \quad \text{Take out the factor } x,$$

$$x(a + b + c) = an. \quad \text{Divide by } a + b + c,$$

$$x = \frac{an}{a+b+c} \text{ or } \frac{n}{a+b+c} \times a = \text{A's share.}$$

$$\frac{bx}{a} = \frac{a}{a+b+c} \times \frac{b}{a} = \frac{bn}{a+b+c} \text{ or } \frac{n}{a+b+c} \times b =$$
B's share.

$$\frac{cx}{a} = \frac{an}{a+b+c} \times \frac{c}{a} = \frac{cn}{a+b+c} \text{ or } \frac{n}{a+b+c} \times c =$$
C's share.

From these formulas we derive the following rule:

149. (1.) *Divide the whole gain (or loss) by the sum of the proportions of the stock; and then multiply this result by the proportion of each;*

(2.) *Divide the product of the whole gain multiplied by each man's proportion of the stock, by the sum of the proportions of the stock.*

Perform the three following examples by this rule.

30. Three men, A, B, and C, trade in company, and gain $1350. Now, if A put into the joint stock $7 as often as B put in $6, and B put in $6 as often as C put in $5, what is each man's share of the gain? – A. 523, B. 450, C, 37

31. Divide 8736 dollars among three men in such a manner, that their shares shall be to each other as the numbers 3, 4, and 5, respectively. A [illegible] B. 2912, [illegible]

32. Two traders, A and B, found that they had gained, at the end of the year, 3792 dollars. A having put $5000 and B $7000 into the joint stock, what is each man's share of the gain? [illegible]

Note.—Their stocks are as 5 to 7.

33. Four towns are situated in the order of the four letters, A, B, C, D. The distance from A to D is 102 miles. The distance from A to B is to the distance from C to D as 2 to 3; and $\frac{1}{4}$ of the distance from A to B, added to $\frac{1}{2}$ the distance from C to D, is 3 times the distance from B to C. How far are the towns apart?

34. Two young men began to trade at the same time, the capital of A being to that of B as 7 to 6. The first year, A gained a sum equal to $\frac{1}{5}$ of his capital, and B lost $\frac{1}{6}$ of his. The second year, A lost $\frac{3}{8}$ of what he then had; and B's gain was to what remained of his original capital, as 2 to 5. At the beginning of the third year, the two had $2450. What was the original capital of each?

$$\text{Let } x = \text{A's capital}$$

$$\frac{6x}{7} = \text{B's} \quad \text{``}$$

$$x + \frac{x}{5} \text{ or } \frac{6x}{5} = \text{A's at the end of the first year}$$

$$\frac{5x}{7} = \text{B's, first year.}$$

$$\frac{6x}{5} - \frac{9x}{20} = \frac{3x}{4} = \text{A's capital, 2d year.}$$

$$\frac{5x}{7} + \frac{2x}{7} = x = \text{B's capital, 2d year.}$$

35. A gentleman, by his will, divided his property equally between his son and daughter; and the son, having spent $\frac{2}{3}$ of his portion, had \$7268 left. How much property had he?

36. A man, being asked the age of his daughter, replied, "My age is to hers as 4 to 1, and her mother's age is to mine as 7 to 8, and the sum of all our ages is 102 years." What was the daughter's age?

37. A trader, having increased his capital by $\frac{1}{4}$ of itself, lost $\frac{1}{3}$ of what he then had; he afterwards gained a sum equal to $\frac{1}{2}$ of the remainder, when he was worth \$3935. What was his capital?

38. The breadth of a certain building is to its height, as 9 to 7; and the height is to the length as 3 to 5; the sum of the length and breadth is 124 feet. What are the dimensions of the building?

Note.—In this example, as the height is compared with both the length and the breadth, it is easier to let $x =$ the height.

State the question in three ways: First, let $x =$ the height, second, let $x =$ the breadth; third, let $x =$ the length.

SECTION IV.

EQUATIONS OF THE FIRST DEGREE.

1. Divide 42 into four such parts that the first shall be 5 more than the second, 8 less than the third, and 9 more than the fourth.

Let $x =$ the first,
$x - 5 =$ the second,
$x + 8 =$ the third,
$x - 9 =$ the fourth.

Uniting the terms $4x - 6 = 42$. Transposing (33), (136).
$4x = 42 + 6 = 48$. Dividing by 4,
$x = 12$.
$x - 5 = 7$; $x + 8 = 20$; and $x - 9 = 3$.
Proof. $12 + 7 + 20 + 3 = 42$.

2. An express had been travelling 5 days, at the rate of 60 miles a day, when another was despatched after him, who travelled 75 miles a day. In how many days did the latter overtake the former?

Let $x =$ the number of days,
Then (5×60) or $300 + 60x =$ No. of miles travelled by the 1st,
$75x =$ do. do. 2d.
$300 + 60x = 75x$. Transposing,
$60x - 75x = -300$. Reducing,
$-15x = -300$.

Changing the signs in both members (which is equivalent to transposing and changing the equation member for member; thus, $300 = 15x$), we have

$+15x = +300$. Dividing.
$x = 20$, the number of days.

3. Let a represent the sum of two members, and b represent their difference. What are the numbers?

Let $x =$ the smaller number.

Then $x + b =$ the larger.

$x + x + b = a$. Transposing and reducing,

$$2x = a - b.$$

$$x = \frac{a-b}{2} \text{ or } \frac{a}{2} - \frac{b}{2}, \text{ the smaller number.}$$

Substitute the value of x to find the larger, thus

$$\frac{a}{2} - \frac{b}{2} + b = \text{the larger. Reducing,}$$

$$\frac{a}{2} - \frac{b}{2} + \frac{2b}{2} = \frac{a}{2} + \frac{b}{2} \text{ the larger.}$$

Stating the formula $\frac{a}{2} + \frac{b}{2}$ and $\frac{a}{2} - \frac{b}{2}$ in words, we have the following rule:

150. *When the* SUM *and the* DIFFERENCE *of two numbers are given;—*

(1.) *To find the* GREATER, ADD *half the difference to half the sum;*

(2.) *To find the* LESS, SUBTRACT *half the difference from half the sum.*

4. The sum of two numbers is 175, and the difference 81, what are the numbers?

Here $a = 175$, and $b = 81$. Substituting in the above formula,

$$\frac{a}{2} + \frac{b}{2} = \frac{175}{2} + \frac{81}{2} \text{ or } \frac{256}{2} = 128, \text{ the larger,}$$

$$\text{and } \frac{a}{2} - \frac{b}{2} = \frac{175}{2} - \frac{81}{2} \text{ or } \frac{94}{2} = 47, \text{ the smaller.}$$

In the same way solve the five following examples.

5. The salaries of two men, A and B, amount to $3529 per

annum; and A receives $721 more than B. What is the salary of each? 1404. 2125

6. A gentleman paid $387 for a horse and chaise; and the chaise cost $75 more than the horse. What was the price of each? 156, 231

7. A man left an estate of $9134, to be so divided between his widow and son, that the former should receive $1486 more than the latter. What was the share of each?

8. A's farm contains 12 acres more than B's; and the two farms together contain 162 acres. Required the number of acres in each.

9. Says A to B, "If you will give me 16 dollars, I shall have as much money as you; and we both have 130 dollars." How much money has each?

10. Three men, A, B, and C, trade in company, upon a capital of $3981; of which B furnished $337 more, and C $181 less, than A. What was the share of each? 1275

11. A man left an estate of $9931, to be divided between his widow, son, and daughter, in such a manner, that the son should have $522 more than the daughter, and $592 less than his mother. Required the portion of each.

12. The water had been flowing from a full cistern 6 hours, at the rate of 12 gallons an hour, when a pipe was conducted into it, which restored 21 gallons an hour. In how many hours was the cistern full again?

Let $x =$ number of hours,
Then $21x =$ gallons by 2d pipe,
$(x + 6)12 =$ gallons by 1st pipe.

13. A father is three times as old as his son; but in 20 years he will be only twice as old. What is the age of each?

14. When a boy would buy a certain number of oranges at

6 cents apiece, he found they would come to 12 cents more than he had; he, therefore, bought the same number at 5 cents each, and had 6 cents left. How many oranges did he buy?

$$\text{Let } x = \text{the oranges,}$$
$$\text{Then } 6x - 12 = \text{his money,}$$
$$\text{and } 5x + 6 = \text{his money.}$$

15. Divide 64 into two such parts, that five times the first shall be equal to three times the second.

$$\text{Let } x = \text{the first part,}$$
$$\text{and } 64 - x = \text{the second.}$$

16. Generalize the above question thus. Divide a into two such parts that m times the first shall be equal to n times the second.

$$\text{Let } x = \text{the first part,}$$
$$\text{and } a - x = \text{the second.}$$
$$\text{Then } mx = n(a - x) = an - nx.$$
$$mx + nx = an.$$
$$x(m + n) = an.$$
$$x = \frac{an}{m+n} \text{ the second.}$$

Substituting the value of x in $a - x$,

$$a - x = a - \frac{an}{m+n}. \quad \text{Reducing (116),}$$

$$a - x = \frac{am + an - an}{m+n} = \frac{am}{m+n}, \text{ the first.}$$

17. Substitute 28 for a, 4 for m, and 3 for n, in the above formula, and find the particular answer to the question.

18. A pole is 4 feet in the ground, $\frac{1}{3}$ of its whole length under water, and $\frac{1}{2}$ above water. Required its length.

19. The head of a fish weighs 8 lbs.; his tail weighs as much as his head and half his body, and his body weighs as much as his head and tail. What is the weight of the fish?

Let $x =$ the weight of the body,

$\frac{x}{2} + 8 =$ the weight of the tail,

and $x + (\frac{x}{2} + 8) + 8 =$ the weight of the fish.

Also, $x = 8 + \frac{x}{2} + 8.$

20. A man, being asked his own and his wife's age, said, that his youngest child was 4 years old; that the age of his wife was twice the age of the child and $\frac{3}{4}$ of his own age; and that his own age was equal to the united ages of his wife and child. How old were they? 44

21. A merchant has wines at 9 shillings, and at 13 shillings, per gallon; and he would make a mixture of 100 gallons, that shall be worth 12 shillings per gallon. How many gallons of each must he take?

$$x = \text{gallons at } 9s.$$
$$100 - x = \text{gallons at } 13s.,$$
$$9x + (100 - x) \times 13 = 100 \times 12.$$

22. Generalize thus: Mix wine at a and at b shillings a gallon, so as to make n gallons at c shillings per gallon.

23. How many gallons of wine, at 9 shillings a gallon, must be mixed with 20 gallons at 13 shillings, that the mixture may be worth 10 shillings a gallon?

24. A merchant, having mixed 10 gallons of wine, at 8 shillings a gallon, with 25 gallons at 10 shillings, wishes to add as much wine at 15 shillings as shall make the whole mixture worth 2 dollars a gallon. How many gallons must he take? See (141).

25. How many gallons of water must be mixed with 35 gallons of wine at 9 shillings, and 45 gallons at 13 shillings, a gallon, that the whole mixture may be worth 10 shillings a gallon?

26. A clerk spends $\frac{2}{3}$ of his salary for board and lodging, $\frac{2}{3}$ of the remainder in clothes, and saves $150 per annum. What is his salary?

27. What is that number, $\frac{1}{3}$ and $\frac{1}{4}$ of which is 35 more than its sixth part?

28. Two men, A and B, have each $80. A spends $5 more than twice as much as B, and has then half as much as B, wanting $13. How much did each spend?

$$\text{Let } x = \text{what B spends,}$$
$$2x + 5 = \text{A spends.}$$
$$80 - x = \text{B has left.}$$
$$80 - (2x + 5) \text{ or } 80 - 2x - 5 = \text{A has left,}$$
$$75 - 2x = \frac{80 - x}{2} - \text{[illegible]}3.$$

29. Divide 84 into two such parts, that if $\frac{1}{2}$ of the less be subtracted from the greater, and $\frac{1}{8}$ of the greater be subtracted from the less, the remainders shall be equal.

$$\text{Let } x = \text{the greater,}$$
$$\text{and } 84 - x = \text{the less.}$$
$$x - \frac{84 - x}{2} = 84 - x - \frac{x}{8}.$$ Clearing of fractions and subtracting (121),

$$8x - 336 + 4x = 672 - 8x - x.$$

151. *In an equation, when the sign — precedes a fraction, the signs in the numerator must be changed, when the denominator is removed.*

The reason of this may be seen by including the numerator in a parenthesis as we remove the denominator; thus, $8x - 4(84 - x) = 8x - 336 + 4x$.

30. Separate 72 into two parts, such that if $\frac{3}{4}$ of the less be subtracted from the greater, the remainder may be equal to $\frac{1}{2}$ the less.

31. Generalize this by separating a into two parts, so that if $\frac{m}{n}$ of the less be subtracted from the greater, the remainder may be equal to $\frac{1}{b}$ the less.

32. Two clerks, A and B, sent ventures in different vessels, A's being worth only $\frac{3}{5}$ as much as B's. A gained and B lost \$23; then $\frac{7}{8}$ of B's returns, subtracted from A's, would leave $\frac{1}{2}$ of the value of A's venture. How much did each send?

33. A gentleman bought a watch and chain for \$160. If $\frac{3}{4}$ of the price of the watch be subtracted from 6 times the price of the chain, and $\frac{5}{12}$ of the price of the chain be subtracted from twice the price of the watch, the remainders will be equal. What was the price of each?

34. A person, being asked the time of day, replied, "If to the time from noon be added its $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, the sum will be equal to the time till midnight." Required the hour.

Let $x =$ the time from noon,
$12 - x =$ the time to midnight.

Note.—Reduce the answer to the fraction of a *minute*.

35. A certain number, when divided by 16, gives such a quotient, that the sum of the dividend, divisor, and quotient, is 84. What is the number?

36. Generalize by substituting a for 16 and b for 84.

37. There are two numbers in the proportion of 3 to 4; but if 24 be added to each of them, the sums will be in the proportion of 4 to 5. What are the numbers?

38. What number is that which, being added to 5, and also multiplied by 5, the product shall be 4 times the sum?

39. A man, having spent \$10 more than $\frac{1}{3}$ of his money, had \$15 more than $\frac{1}{2}$ of it left. How much had he?

40. Divide 72 into two unequal numbers, so that, if $\frac{3}{4}$ of the less be subtracted from the greater, the remainder may be equal to $\frac{2}{3}$ of the number which remains, after the excess of the greater above the less is subtracted from the less.

$$x = \text{the less,}$$

$$72 - x = \text{the greater,}$$

$$(72 - x - x) \text{ or } 72 - 2x = \text{excess of greater above the less.}$$

41. If A can dig a certain trench in a days, and B can do it in b days, in how many days will they both do it?

Let x = the number of days,

then $\frac{1}{x}$ = what both do in one day.

But $\frac{1}{a} + \frac{1}{b}$ = also what both do in one day. Hence,

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{x}.$$

Note.—Interpret the formula in words, and perform the following example by it.

42. If A can dig a well in 16 days, and B can dig it in 20 days, how many days will it take both to do it?

SECTION V.

EQUATIONS OF THE FIRST DEGREE.

1. What number is that, to which if there be added $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the sum will be 50?

2. A person, upon being asked his age, replied, that $\frac{1}{3}$ exceeded $\frac{1}{4}$ part of it by 5 years. What was his age?

3. A trader gave three checks, amounting to $94; the first

for $\frac{1}{3}$, the second for $\frac{1}{4}$, and the third for $\frac{1}{5}$, of the money he had in the bank. How much had he?

4. Find two numbers, in the ratio of 2 to 1, such that if 4 be added to each, they will be in the ratio of 3 to 2.

5. In the composition of a certain quantity of gunpowder, $\frac{2}{3}$ of the whole and 10 pounds were nitre; $\frac{1}{6}$ of the whole, wanting $4\frac{1}{2}$ pounds, was sulphur; and the charcoal was $\frac{1}{7}$ of the nitre, wanting 2 pounds. How many pounds were there?

NOTE.—The sum of the pounds of nitre, of sulphur, and of charcoal, must be equal to the whole number of pounds in the gunpowder.

6. A person, being asked the time of day, answered, that the time past from noon was equal to $\frac{4}{5}$ of the time to midnight What was the hour?

7. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off, making way at the rate of 8 miles an hour. In how many hours will the ship be overtaken?

8. Required two numbers, which are to each other as 3 to 2; and whose sum equals $\frac{1}{6}$ part of their product.

$$\text{Let } x = \text{the larger,}$$

$$\frac{2x}{3} = \text{the smaller,}$$

$$x + \frac{2x}{3} \text{ or } \frac{5x}{3} = \text{sum,}$$

$$\frac{2x^2}{3} = \text{product,}$$

$$\frac{5x}{3} = \frac{x^2}{9}.$$

This is an equation of the 2d degree, but as x is a factor in every term, dividing by x, we have $\frac{5}{3} = \frac{x}{9}$, an equation of the first degree.

9. Generalize the preceding question, by supposing the larger to be to the smaller as m to n, and their sum to be $\frac{1}{b}$ part of their product.

10. Divide 48 into two such parts that the excess of the greater above 20, may be three times the excess of 20 above the less.

11. A father leaves $1600, to be divided between his widow, son and daughter, in such a manner that the widow is to have $200 more than the son, and the son $100 more than the daughter. What is the share of each?

12. A man leaves $11,000, to be divided between his widow, two sons, and three daughters. By his will the mother is to receive twice as much as one of the sons; and each son is to receive twice as much as a daughter. How much is each of them to receive?

13. An estate is divided between three men in such a manner, that A receives $1000 less than $\frac{1}{2}$, B $800 less than $\frac{1}{3}$, and C $600 less than $\frac{1}{4}$ of the whole. What is the value of the whole estate, and what is the share of each individual?

14. A father leaves his property to four sons, who share it in the following manner: A has $3000 less than $\frac{1}{2}$; B has $1000 less than $\frac{1}{3}$; C has $\frac{1}{4}$; and D has $600 more than $\frac{1}{5}$ of the property. What is the whole amount bequeathed? and the share of each of the sons?

15. Divide 76 into two such parts, that the quotient of the greater, divided by the less, may be 37.

Let $x =$ the less number,
then $76 - x =$ the greater.

16. Divide a into two such parts, that the quotient of the greater by the less, may be b.

17. A man has two silver cups, having but one cover for both. The first cup weighs 12 ounces; and when it is covered, it weighs twice as much as the other cup; but if the second cup be covered, it weighs three times as much as the first. Required the weight of the cover and of the second cup.

Let $x =$ weight of the cover,

$\frac{x+12}{2} =$ weight of second cup.

18. If a certain number be divided by 7, the sum of the dividend, divisor, and quotient, will be 71. What is the number?

19. What is that number, of which $\frac{1}{7}$ is as much smaller than 65, as twice the number is greater than 640?

20. Two persons, A and B, are 320 miles apart, and travel towards each other; A at the rate of 9 miles an hour, and B at the rate of 7 miles. In what time will they meet, supposing them to start at the same time? How many miles does each travel?

21. Two brothers, A and B, had the same annual income. A spent all of his, and $\frac{1}{7}$ more; B saved $\frac{1}{5}$ of his. At the end of 10 years, B paid A's debts, and had $160 left. What was their income?

NOTE.—A's annual debt is $\frac{x}{7}$.

22. A farmer planted corn in 4 fields. The third produced 9 bushels more than the fourth; the second, 12 bushels more than the third; the first, 18 bushels more than the second; and the whole produced 6 bushels more than 7 times as much as the fourth. What was the whole number of bushels?

23. Two men have equal sums of money. One having spent $39, and the other $93, the one has but half as much left as the other. How much had each?

24. What is that number, which being added to 24 and to 36, the sums will be to each other as 7 to 9?

25. A laborer receives 3*s.* 6*d.* every day he works, and forfeits 9*d.* every day he is idle. At the end of 24 days, there is due to him the sum of £3 2*s.* 9*d.* How many days was he idle?

Let $x =$ number of days he was idle,
and $24 - x =$ number of days he worked.

Also see (141).

26. A laborer was employed for n days, with the understanding that he should receive b shillings for every day he worked, and forfeit c shillings for every day he was idle. At the end of the time, he received a shillings. How many days was he idle?

27. Two persons, A and B, have each an annual income of $400. A spends, every year, $40 more than B; and, at the end of 4 years, they both together save a sum equal to the income of either. What do they spend annually?

28. A gentleman leaves $315, to be divided among four servants in the following manner: B is to receive as much as A, and $\frac{1}{2}$ as much more; C is to receive as much as A and B, and $\frac{1}{3}$ as much more; D is to receive as much as the other three, and $\frac{1}{4}$ as much more. What is the share of each?

29. What number is that, which being multiplied by 4, and 30 subtracted from the product, and being divided by 4, and 30 added to the quotient, the sum and difference shall be equal?

30. A person, being asked his age, replied, "If $\frac{3}{8}$ of my age be multiplied by $\frac{1}{9}$ of my age, the product will be equal to my age." What was his age?

31. Two numbers are to each other as 2 to 3: but if 50 be

subtracted from each, one will be $\frac{1}{2}$ of the other. What are the numbers?

32. In a mixture of wine and cider, $\frac{1}{2}$ of the whole and 25 gallons was wine; and $\frac{1}{3}$ of the whole, wanting 5 gallons, was cider. Required the quantity of each in the mixture.

33. In a certain river is a post that stands 4 feet in the ground; at high water, $\frac{5}{6}$ of the remainder is covered, while only $\frac{1}{8}$ of the whole post appears above the surface of the water. Required the length of the post.

34. What number is that, to which if I add 13, and from $\frac{1}{13}$ of the sum subtract 13, the remainder shall be 13?

35. Three men, A, B, and C, build 318 rods of wall: A builds 7 rods, B 6 rods, and C 5 rods a day: B works twice as many days as A, and C works $\frac{1}{2}$ as many days as both A and B. How many days does each work?

36. A farmer has his sheep in four pastures: in the first, $\frac{1}{3}$ of his flock; in the second, $\frac{1}{4}$; in the third, $\frac{1}{6}$; and in the fourth, 18 sheep. How many sheep has he?

37. If a man fills a certain chest with corn, at 5*s.* a bushel, he will spend all his money; but if he fills it with oats, at 3*s.* 6*d.* a bushel, he will have £1 4*s.* left. How many bushels does the chest hold?

38. Three merchants, A, B, and C, engage in a speculation, by which they gain $960. A put in $3 as often as B $7, and C $5. What is each man's share of the gain?

NOTE.—See Ex. 29, Section III.

39. Says John to William, "I have three times as many marbles as you." "Yes," says William; "but if you will give me 20, I shall have 7 times as many as you." How many has each?

40. A woman sells eggs and apples; the eggs are worth 5 cents a dozen more than the apples; and 8 dozen of eggs are worth as much as 13$\frac{5}{7}$ dozen of her apples. What is the price of each?

41. Two travellers found some five-dollar bills in the road, of which A secured twice as many as B; but had B secured 5 more of the bills, he would have had 3 times as much money as A. How many dollars did each find?

42. A man's age, when he was married, was to that of his wife as 6 to 5; and after they had been married 8 years, her age was to his as 7 to 8. What were their ages when they were married?

43. A gentleman gave $44 more for his chaise than for his horse. Now, if $\frac{4}{7}$ of the price of the horse be subtracted from the price of the chaise, the remainder will be the same as if $\frac{2}{3}$ of the excess of the price of the horse above $84 be subtracted from the price of the horse. What did he give for the horse?

44. A man and his wife usually drank a cask of beer in 12 days; but when the man was from home, it lasted his wife 30 days. How many days would it last the man alone?

Let x denote the days it would last the man alone,

then $\frac{1}{x}$ = what he consumes in one day,

$\frac{1}{30}$ = what his wife consumes in a day,

$\frac{1}{12}$ = what both consume in a day. Hence,

$$\frac{1}{12}=\frac{1}{x}+\frac{1}{30}.$$

45. A and B together can build a piece of wall in 8 days; and, with the assistance of C, they can build it in 5 days. In how many days could C build it alone?

46. Generalize this, by supposing A to build the wall alone in a days; and that A and B build it in n days. How many days will it take B alone?

47. A cistern has two cocks, one of which will empty it in 7 hours, the other in 9 hours. How long will it take both to empty it?

48. If a reservoir can be exhausted by one engine in 7 hours, by another in 8 hours, and by a third in 9 hours, in what time will it be exhausted, if they are all worked together?

49. A reservoir can be filled by two hose companies in 12 hours, and by one of them alone in 20 hours. In what time could the other fill it?

50. In an orchard of fruit-trees, $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ of them pears, $\frac{1}{8}$ of them peaches, 7 trees bear cherries, 3 plums, and 2 quinces. How many trees are there in the orchard?

51. A boy, being asked his age, answered, that if $\frac{1}{2}$ and $\frac{1}{4}$ of his age and 20 years more were added to his age, the sum would be three times his age. How old was he?

52. A father is 40 years old, and his son is 8. In how many years will the father's age be three times the son's?

53. Two travellers, A and B, find a purse with dollars in it. A takes out \$2 and $\frac{1}{3}$ of what remains, and B takes out \$3 and $\frac{1}{3}$ of what remains; when they have equal shares. How much money did they find?

54. A grocer bought a quantity of oats at the rate of 2 bushels for a dollar, and as many more at the rate of 3 bushels for a dollar; and he sold them 5 bushels for 3 dollars, by which he gained \$10. How many bushels did he sell?

Let $x =$ number of bushels of each kind,

$$\frac{x}{2} + \frac{x}{3} = \text{price he gave,}$$

$$2x \text{ bushels at } \$\,\frac{3}{5} = \frac{6x}{5} = \text{price he sold it for.}$$

55. Two clerks, A and B, have the same income. A saves $\frac{1}{5}$ of his; but B, by spending \$80 a year more than A, at the end of 4 years finds himself \$220 in debt. What was their income?

A *spends* annually $\frac{4x}{5}$.

56. After spending $\frac{1}{4}$ of my money, and $\frac{1}{5}$ of the remainder, I had \$96 left. How much had I at first?

57. A traveller spends $\frac{1}{3}$ of his money in Boston; $\frac{1}{7}$ of the remainder in Providence; $\frac{4}{9}$ of what was left in New York; $\frac{3}{5}$ of the balance in Philadelphia, and had \$80 left. How much had he at first?

58. Divide 26 into three such parts, that, if the first be multiplied by 2, the second by [illegible] and the third by 4, the products shall all be equal.

59. Divide 56 into two such parts, that, the larger x being divided by 7, and the smaller by 3, the sum of their quotients may be 10. $56 - x = $ ″ $\frac{x}{7} + \frac{56 - x}{3} = 10$

$3x + 392 - 7x = 210$

$-4x = -182$

$x = 45$

60. A cistern has three cocks; the first will fill it in 5 hours, the second in 10 hours, and the third will empty it in 8 hours. In what time will the cistern be filled, if all the cocks are running together?

Let $x =$ the number of hours required to fill it when all the cocks are running.

$\frac{1}{x} =$ the quantity in the cistern in one hour,

$\frac{1}{5} =$ the part of the cistern filled by the first in an hour,

$\frac{1}{10} =$ the part filled by the second in an hour,

$\frac{1}{8} =$ the part emptied by the third in an hour.

$$\frac{1}{5} + \frac{1}{10} - \frac{1}{8} = \frac{1}{x}.$$

$5x = 40$

$7x = 40$

$x = 5\frac{5}{7}$

61. A school-boy, being asked his age, replied, that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age, would give a product equal to his age. How old was he?

$\frac{3x^2}{48} = x$

62. A person has a lease for 99 years; and, being asked how much of it had expired, he replied, that $\frac{2}{3}$ of the time past was equal to $\frac{4}{5}$ of the time to come. How many years had the lease run?

$3x^2 = 4$

$x = 1$

$x =$ time past it had run,

$99 - x =$ time it had to run.

63. What number is that which may be divided into either two or three equal parts, and the continued product of those parts shall be equal?

$$\frac{x}{2} \times \frac{x}{2} = \frac{x}{3} \times \frac{x}{3} \times \frac{x}{3}.$$

64. A shepherd, driving a flock of sheep in time of war, meets with a company of soldiers, who plunder him of half his flock and half a sheep over; and a second, third, and fourth company treat him in the same manner, each taking half the flock left by the last and half a sheep over, when but 8 sheep remained. How many sheep had he at first?

65. A gentleman has two horses, and a chaise worth $150. Now, if the first horse be harnessed, the horse and chaise together will be worth twice as much as the second horse; but if the second horse be harnessed, they will be worth three times as much as the first horse. What is the value of each horse?

66. Divide 54 into two such parts, that, if the greater be divided by 9, and the less by 6, the sum of the quotients shall be 7.

67. A farmer sells a quantity of corn, which is to the quantity left as 4 to 5. After using 15 bushels, he finds he has $\frac{1}{2}$ as much left as he sold. How many bushels had he at first?

68. Divide 84 into two such numbers, that the quotient of the greater, divided by their difference, may be four.

69. A laborer agreed to work for a gentleman a year for \$72 and a suit of clothes; but, at the end of 7 months, he was dismissed, having received his clothes and \$32. What was the value of the clothes?

Let $x =$ the value of the clothes,

then $72 + x =$ his yearly wages,

and $6 + \frac{x}{12} =$ his wages per month.

70. A laborer reaps 35 acres of wheat and rye. For every acre of rye he receives 5 shillings; and what he receives for an acre of wheat, if it were one shilling more, would be to what he receives for an acre of rye as 7 to 3. For the whole he receives £13. How many acres are there of each sort?

71. A man being asked how much money he had, replied, "If you multiply my money by 4, add 60 to the product, divide the sum thus obtained by 3, and then subtract 45 from the quotient, the remainder will be the number of dollars I have." How much money had he? x = money

$\frac{4x + 60}{3} - 45 = x$

$4x + 60 - 135 = 3x$

$4x - 3x = 135 - 60$

$x = 75$

70. x = acres in wheat

35 - x = " " rye

$\frac{7}{3}$ of 5 $= \frac{35}{3} - 1 = \frac{32}{3}$

$(35 - x)5 + \frac{32x}{3}$ [illegible]

[illegible]

[illegible] $x + 32x = 780$

$17x$ [illegible] $= 780 - 525$

[illegible]

$x = 15$

$35 - x = 20$

CHAPTER V.

EQUATIONS OF THE FIRST DEGREE WITH TWO OR MORE UNKNOWN QUANTITIES.

SECTION I.

TWO UNKNOWN QUANTITIES.

ALL the examples given heretofore, have contained only *one* unknown quantity. There are many cases where it is convenient, and others where it is necessary, to employ *two* unknown quantities.

Where two unknown quantities are employed, there must always be two distinct equations. Then to solve these equations,

152. *From the two equations, containing two unknown quantities,* ONE *equation must be derived, containing but* ONE *unknown quantity.*

The unknown quantity that is made to disappear, is said to be *eliminated*. There are three methods of elimination, with all of which the learner should become familiar, as it is sometimes convenient to use them all, in solving the same problem.

Elimination by comparison.

1. A gentleman has two silver cups, and a cover adapted to each which is worth £10. If the cover be put upon the first cup, its value will be twice that of the second; but if the cover be put on the second, its value will be three times that of the first. What is the value of each cup?

Let $x =$ the value of the first cup,
and $y =$ the value of the second,

Then (1) $10 + x = 2y$,
and (2) $10 + y = 3x$.

If now we solve these equations, to obtain the value of x, instead of its being found in terms of *known* quantities (133), it will be found to contain terms of an *unknown* quantity, y.

Thus, (3.) $x = 2y - 10$, from equation (1),

(4.) $x = \frac{10 + y}{3}$, from equation (2).

Since things which are equal to the same thing are equal to each other (16), we may put these values of x equal to each other, thus eliminating x and obtaining a single equation, containing only one unknown quantity. Then,

(5.) $2y - 10 = \frac{y + 10}{3}$. Multiplying by 3,

(6.) $6y - 30 = y + 10$. Transposing,

$6y - y = 10 + 30$. Reducing,

$5y = 40$. Dividing by 5,

$y = 8$, the price of the second.

The value of x may be found by substituting the value of y in equation (3) or (4). Substituting in (3) we have,

$$x = 2 \times 8 - 10 = 16 - 10, \text{ hence}$$

$$x = 6, \text{ the price of the first.}$$

By this method of elimination,—or *elimination* by comparison;

153. *The value of one of the unknown quantities is found in each of the equations, and then a new equation is formed, containing only the other unknown quantity, by comparing these values together.*

2. There is a certain fraction, such that if 7 be added to the numerator, the value of the fraction becomes 2; if 7 be added to its denominator, it becomes $\frac{1}{4}$. What is the fraction?

Let $x =$ the numerator,
and $y =$ the denominator,
then $\frac{x}{y} =$ the fraction.

We have by the conditions of the question,

$$(1.)\quad \frac{x+7}{y}=2,$$

$$(2.)\quad \frac{x}{y+7}=\tfrac{1}{4}.$$ Clearing of fractions,

$$(3.)\quad x+7=2y.$$

$$(4.)\quad 4x=y+7.$$

$$(5.)\quad y=\frac{x+7}{2}.$$

$$(6.)\quad y=4x-7.$$

$$(7.)\quad 4x-7=\frac{x+7}{2}.$$

$$8x-14=x+7.$$

$$7x=21.$$

$$x=3.$$ Substituting this value of x in equation (5), we shall have,

$$y=\frac{3+7}{2}=\frac{10}{2}=5.$$

$$\frac{x}{y}=\tfrac{3}{5},$$ the required fraction.

In these examples, as in all others, the first operation should be to clear the equations of fractions. The learner should exercise his judgment as to *which* quantity can be the most readily eliminated.

3. There are two numbers whose sum is 120; and if 4 times the less be subtracted from 5 times the greater, the remainder will be 150. Required the numbers.

4. If the greater be added to half the less of two numbers, the sum is 48; but if the less be added to half the greater, the sum is 42. What are the numbers?

5. A vintner has two sorts of wine, which, if mixed in equal parts, will be worth 15 shillings a gallon; but if 2 gallons of

the first be mixed with 3 gallons of the second, a gallon of the mixture will be worth only 14 shillings. What is each sort worth per gallon?

Let $x =$ the price of the first per gallon,
and $y =$ the price of the second.
then $x + y = (15s. \times 2) = 30s.$,
and $2x + 3y = (14s. \times 5) = 70s.$

6. It is required to find two numbers, such that $\frac{1}{3}$ of the first and $\frac{1}{4}$ of the second shall be 87, and $\frac{1}{5}$ of the first and $\frac{1}{6}$ of the second shall be 55. ✓

7. Says A to B, "6 years ago, your age was double mine; and, in 4 years, my age will be $\frac{3}{5}$ of yours." What is the age of each? ✓

8. There is a number consisting of two figures or digits, and if the number be divided by the sum of the digits, the quotient is 4; but if the digits be inverted, and the number divided by 1 more than their sum, the quotient will be 6. What is the number?

Let $x =$ the first digit,
and $y =$ the second digit.

Then, since the first digit is in the place of tens, we shall have, $10x + y =$ the number.

Now by the conditions of the question,

$$\frac{10x + y}{x + y} = 4.$$

$$\frac{10y + x}{x + y + 1} = 6.$$

9. A farmer sold to one man 10 bushels of corn and 12 bushels of potatoes for 54 shillings; and to another, 2 bushels of corn and 4 bushels of potatoes for 14 shillings. What was the price of each per bushel?

10. A gentleman gave to his two sons, A and B, 9800

dollars. At the end of a year, A finds that he has spent $\frac{1}{5}$ of his share; but B, having spent only $\frac{1}{6}$ of his, has just as much left as his brother. What was the share of each?

11. Says A to B, "Give me 6 dollars, and I shall have 4 times as much as you." "Rather give me 3 dollars," says B, "and I shall have just as much as you." How many dollars has each?

12. If I take 10 apples from A, he will still have twice as many as B; but if I give them to B, they will each have the same number. How many have they?

SECTION II.

ELIMINATION BY SUBSTITUTION.

1. A builder paid 5 men and 3 boys 42 shillings for working a day; he afterwards hired 7 men and 5 boys, for 62 shillings a day. What were the wages of each?

Let $x =$ the wages of a man,
and $y =$ the wages of a boy.

Then (1.) $5x + 3y = 42,$
and (2.) $7x + 5y = 62.$

Now if we find the value of x in the first equation, we may substitute that value in the second, and thus from the two equations, deduce one with one unknown quantity (152).

(3.) $x = \frac{42 - 3y}{5}.$ Substituting this value,

(4.) $\frac{7(42 - 2y)}{5} + 5y = 62.$

(5.) $294 - 21y + 25y = 310.$
$-21y + 25y = 310 - 294.$
$4y = 16$, and $y = 4.$

Substituting in equation (3) we shall have,

$$x = \frac{42 - 12}{5} = \frac{30}{5} = 6.$$

Ans. 4*s.* boy's wages; 6*s.* man's wages.

In this method of elimination,—or *elimination* by substitution,

154. *The value of either of the unknown quantities, is found in one of the equations, as if the other quantity were known; and then this value is substituted in the other equation instead of the quantity itself.*

2. If a certain room were 5 ft. broader and 3 ft. longer, the floor would contain 422 square ft. more than it does; but if it were 3 ft. broader and 5 ft. longer, it would contain 400 square ft. more. What are the dimensions of the room?

Let $x =$ the length,

and $y =$ the breadth.

then $xy =$ the area, or number of sq. ft. [in the floor.

Now (1.) $(y + 5)(x + 3) = xy + 422.$

and (2.) $(y + 3)(x + 5) = xy + 400.$

(3.) $xy + 5x + 3y + 15 = xy + 422.$

(4.) $xy + 3x + 5y + 15 = xy + 400.$ Transposing,

(5.) $3x = 400 - 15 - 5y.$

(6.) $x = \frac{385 - 5y}{3}.$

$$5 \times \frac{385 - 5y}{3} + 3y = 422 - 15.$$ Substituting

in (3), $\frac{1925 - 25y}{3} + 3y = 407.$

$$1925 - 25y + 9y = 1221.$$

$$-25y + 9y = 1221 - 1925.$$

$$-16y = -704.$$

$y = 44$ ft. the breadth of the room;

$$x = \frac{385 - 5 \times 44}{3} = \frac{385 - 220}{3} = \frac{165}{3} = 55,$$ the length.

3. Says A to B, "Give me \$8, and I shall have twice as much money as you will have left; but if I give you \$6, my money will be equal to but $\frac{1}{4}$ of yours." How much has each?

4. A gentleman has two horses, and a chaise worth \$250. If the first be harnessed, the horse and chaise will be worth twice as much as the second horse; but if the second be harnessed, they will be worth three times as much as the first horse. What is the value of each horse?

5. A merchant bought two lots of flour for \$576; the first lot for \$5, and the second for \$6, per barrel. He then sold $\frac{5}{8}$ of the first lot and $\frac{4}{7}$ of the second for \$353, by which he gained \$11. How many barrels were there in each lot? See (141).

6. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{1}{2}$; but the denominator being doubled, and the numerator increased by 7, the value becomes 1?

7. If 6 feet were added to each side of a hall, the breadth would be to the length as 6 to 7; but if 6 feet were taken from each of the sides, they would be to each other as 4 to 5. Required the dimensions of the hall.

8. A farmer has 86 bushels of wheat at 4*s.* 6*d.* a bushel, with which he wishes to mix rye at 3*s.* 6*d.* a bushel, and barley at 3*s.* a bushel, so as to make 136 bushels, that shall be worth 4*s.* a bushel. How much rye and barley must he take?

Let $x =$ number of bushels of rye,
and $y =$ number of bushels of barley.

Then $x + y + 86 = 136$,
and $86 \times 4\frac{1}{2} + 3\frac{1}{2}x + 3y = 136 \times 4$.

See (141). [illegible]

9. If you multiply the greater of two numbers by 3 and the less by 4, the difference of their products is 48; but if you divide the greater by 4 and the less by 3, the sum of their quotients will be 14. What are the numbers?

10. A gentleman, having a quantity of gold and silver coins, finds that 24 pieces of gold and 40 pieces of silver will pay a certain debt; of which 5 pieces of gold and 15 pieces of silver will pay $\frac{1}{4}$ part. How many pieces of gold, and how many of silver, will pay the whole debt?

$x =$ the value of a piece of gold,
$y =$ the value of a piece of silver.

$$24\cancel{x} + 40\cancel{y} = 1, \text{ (that is, one debt,)}$$
$$5\cancel{x} + 15\cancel{y} = \tfrac{1}{4}.$$

11. There is a certain number consisting of two digits whose sum is 6. If 18 be added to the number, the sum will consist of the same digits inverted. What is the number?

12. The area of a certain garden is 128 square rods; and if the garden were 4 rods longer, it would contain an acre. Required the length and width.

NOTE.—An Acre = 160 square rods.

SECTION III.

TWO UNKNOWN QUANTITIES.

Elimination by Addition and Subtraction.

1. A man bought 3 bushels of wheat and 5 bushels of rye for 38 shillings; he afterwards bought 6 bushels of wheat and 3 of rye for 48 shillings. What did he give a bushel for each?

Let $x =$ the price of the wheat per bushel,
and $y =$ the price of the rye.

Then (1.) $3x + 5y = 38$.
and (2.) $6x + 3y = 48$.

Now examining the coefficients of the unknown quantities in equations (1) and (2), we find that if we multiply $3x$ by 2, it will have the same coefficient with x in equation (2). Multiplying equation (1) by 2, we shall have,

$$\begin{array}{ll} (3.) & 6x + 10y = 76. \\ (2.) & \underline{6x + \ \ 3y = 48.} \quad \text{Subtracting equation (2),} \\ & \qquad\quad 7y = 28. \\ & \qquad\quad y = 4, \text{ the price of the rye.} \end{array}$$

Substituting the value of y in equation (1), we shall have,

$$3x + 5 \times 4 = 38,$$

$$3x = 38 - 20 = 18,$$

$$\text{and } x = 6, \text{ the price of the wheat.}$$

Since the signs of $6x$ are both $+$, we *subtract* the less equation (which is always determined by the known quantity in the second member) from the greater. Changing the signs, we have,

$$6x + 10y - 6x - 3y = 76 - 48, \text{ or, as above,}$$

$$7y = 28.$$

In subtracting, the signs may be changed mentally, but great care is necessary in this part of the operation.

2. A boy bought 7 oranges and 5 lemons for 55 cents; and afterwards sold, at the same rate, 4 oranges and 3 lemons for 32 cents. What was the price of each?

Let x = the price of an orange,
and y = the price of a lemon.

Then (1.) $7x + 5y = 55$,
and (2.) $4x + 3y = 32$.

Examining the original equations, we find that the coefficients of both x and y are prime to each other, but as the coefficients of y are the smaller, we will eliminate y.

$$\begin{array}{ll} (3.) & 21x + 15y = 165. \quad \text{Multiplying equation (1) by 3.} \\ (4.) & \underline{20x + 15y = 160.} \quad \text{Multiplying equation (2) by 5.} \\ & \qquad\quad x = 5. \quad\ \ \text{Subtracting equation (4).} \end{array}$$

$4 \times 5 + 3y = 32.$ Substituting the value of x in equation (2), $3y = 32 - 20 = 12.$

$y = 4$ cts., price of the lemons.

We substitute in equation (2), as the coefficients are simpler.

3. Two boys, A and B, talking together, says A to B, "6 times my money added to 8 times yours, is $52; but 3 times my money and 7 times yours, will amount to $32." How much had each?

Let $x =$ A's money,

$y =$ B's money.

(1.) $6x + 8y = 52.$

(2.) $3x + 7y = 32.$

It is evident on inspecting these equations, that it will be easier to eliminate x than y. This may be done by *dividing* equation (1) by 2,

(3.) $3x + 4y = 26.$ Dividing equation (1) by 2.

(2.) $3x + 7y = 32.$

$3y = 6.$ Subtracting equation (3).

$y = 2$, B's money.

Substituting in equation (3),

$3x + 8 = 26$; $3x = 26 - 8 = 18,$

$x = 6$, A's money.

4. If twice A's money be subtracted from 3 times B's, the remainder is $38; but if twice B's money be subtracted from 3 times A's, the remainder is $83. How much has each?

Let $x =$ A's money,

$y =$ B's money.

(1.) $3y - 2x = 38,$

(2.) $3x - 2y = 83.$

(3.) $-6x + 9y = 114.$ Multiplying equation (1) by 3.

(4.) $6x - 4y = 166.$ Multiplying equation (2) by 2.

$5y = 280.$ Adding equations (3) and (4).

Then $y = 56$, B's money.

Substituting in equation (1), $3 \times 56 - 2x = 38$.

$$168 - 38 = 2x;\ 2x = 130.$$

$$x = 65, \text{ A's money.}$$

In equation (3) we made the terms containing x and y change place, for the sake of having the similar terms under each other. Since the signs of $6x$ are unlike, these terms will cancel when *added*. We have then the following rule, for this method of elimination:

155. *Having determined which of the unknown quantities you will eliminate, make the coefficients of the terms, containing that quantity, the same in both equations, either by multiplication or division.*

If the signs of these terms are unlike, add both equations together; if alike, subtract the smaller from the larger equation.

5. Says A to B, "$\frac{1}{3}$ of the difference of our money is equal to yours; and, if you give me \$2, I shall have 5 times as much as you." How much has each?

6. Required the number, from which if 27 be subtracted, the digits of which it is composed will be inverted; the sum of the digits being 9.

7. A man has money in two drawers, and \$25 in his purse. Now, if he put his purse into the first drawer, it will contain $\frac{5}{7}$ as much as the second; but if he put his purse into the second drawer, it will contain $\frac{13}{5}$ as much as the first. How much is in each drawer?

8 Two clerks, A and B, sent ventures, by which A gained \$20, and B lost \$50, when the former had twice as much as the latter; but had B gained \$20, and A lost \$50, then B would have had 4 times as much as A. What sum was sent by each?

9. A farmer, having mixed a certain quantity of barley and oats, found that, if he had mixed 6 bushels more of each, he would have put into the mixture 7 bushels of barley for every 6 of oats; but if he had mixed 6 bushels less of each, he would have put in 6 bushels of barley for every 5 of oats. How many bushels did he mix?

Note.—That is, in the first case, they would have been to each other as 7 to 6.

10. A person has a gold watch and a silver one, and a chain for both worth $8. Now, the silver watch and chain are together worth half as much as the gold watch; but when the chain is on the gold watch, they are together worth three times as much as the silver watch. What is the value of each?

11. If a certain volume contained 12 more pages, with 3 lines more upon a page, the number of lines would be increased by 744; but if it contained 8 pages less, and the lines on a page were not so many by 4, the whole number of lines would be diminished by 680. How many pages are there in the book? and how many lines on a page?

Note.—xy will represent the number of lines in the book.

12. Two neighbors, A and B, possess 562 acres of land. If A's farm were 4 times, and B's 3 times, as large as each of them is, they would both together have 1924 acres. How many acres has each?

13. Two men owe more money than they can pay. Says A to B, "Give me $\frac{1}{5}$ of your property, and I shall be able to pay my debts." "If you will give me $\frac{1}{4}$ of yours," replies B, "I shall be able to pay my own." The amount of A's debts is $1500, and of B's, $2125. How much property has each in his possession?

14. A trader bought at auction two pipes containing wine. For one he gave 8*s.* a gallon; for the other, 10*s.* 6*d.*; and the whole came to £48. Having sold 25 gallons from the first

pipe, and 16 gallons from the second, he mixed the remainders together, and added 15¾ gallons of water. Afterwards, 5¾ gallons of the mixture leaked out; and the remainder was worth 8*s.* a gallon. How many gallons did each pipe contain?

The following examples may be solved by any one of the preceding methods of elimination, which best meets the nature of the case.

15. Required two such numbers, that, if ⅓ of the first be added to ⅚ of the second, the sum shall be 66; but if ⅚ of the first be added to ⅓ of the second, their sum shall be 60.

16 Two persons, A and B, talking of their ages, says A to B, "12 years ago I was twice as old as you; and in 12 years my age will be to yours as 3 to 2." What is the age of each?

17. A vintner sold to one man 16 dozen of sherry wine and 19 dozen of port, for $382; and to another man, 24 dozen of sherry and 17 dozen of port, for $458;—the prices being the same to both. What was the price of each kind of wine?

18. If you add 2 to the numerator of a certain fraction, its value becomes ½; but if you add 2 to its denominator, the fraction will be equal to ¼. What is the fraction?

19. The fore-wheel of a coach makes 5 revolutions while the hind-wheel is making 4; but if the circumference of each were one yard greater, their revolutions would be to each other as 6 to 5. What is the circumference of each in feet?

20. A gentleman gave $4350 for a house-lot, the land being valued at $2 a foot. If it had been 6 feet wider, it would have cost $5394. What were the length and breadth of the lot?

21. I have a certain number of cents in each hand. If I put 10 out of my left hand into my right, there will be twice

as many in my right as remain in my left; but if I put 10 out of my right hand into my left, there will be three times as many in my left hand as remain in my right. How many cents have I in each hand?

22. There are two numbers such that if you multiply the greater by 2 and the less by 3, the sum of their products is 101. And if you divide the greater by 4 and the less by 5, the sum of their quotients is 10. Required the numbers.

23. If you divide the greater of two numbers by the less, the quotient will be 7; and the amount of the numbers is 1008. Required the numbers

24. Two men, A and B, are employed to set up 220 rods of fence. If A work 9 days and B 8, the fence will not be completed by 2 rods; but if A work 8 days and B 9, they will be able to finish the fence and 4 rods more. How many rods can each build in a day?

25. A man had 32 gallons of wine, in two barrels. Wishing to have an equal quantity in each, he poured out of the first into the second as much as the second already contained; again, he poured out of the second into the first as much as it then contained; and, finally, he poured out of the first into the second as much as still remained in it. Each barrel then contained the same quantity. How many gallons did they contain originally?

SECTION IV.

EQUATIONS CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

156. *When more than two unknown quantities are required, there must always be as many equations as there are unknown quantities.*

These equations are solved on the same principles as have been already given; that is, we eliminate the quantities, one by one, until we finally obtain a single equation with but one unknown quantity.

Thus, if there are *four* unknown quantities, there must also be four equations. From these four equations, we obtain, by eliminating some one of the unknown quantities, *three* equations, with *three* unknown quantities. From these we deduce *two* equations, with *two* unknown quantities; from which again we obtain *one* equation with *one* unknown quantity.

A great deal of care and judgment is required in selecting the quantity to be first eliminated, and also the method of elimination best adapted to the case. Practice will soon enable the pupil to choose the shortest and simplest way of solving the equations.

1. Three boys, A, B, and C, bought fruit at the same time. A bought 4 oranges, 7 peaches, and 5 pears for 51 cents; B bought 6 oranges, 8 peaches, and 10 pears, for 74 cents; and C bought 9 oranges, 3 peaches, and 2 pears, for 58 cents. What was the price of each?

Let $x =$ the price of an orange,
$y =$ the price of a peach,
and $z =$ the price of a pear.

(1.) $4x + 7y + 5z = 51.$
(2.) $6x + 8y + 10z = 74.$
(3.) $9x + 3y + 2z = 58.$

Upon inspection, it will be seen that to eliminate z by the third method, will involve the fewest operations upon the equations. Dividing equation (2) by 2, we shall have,

(4.) $3x + 4y + 5z = 37$
(1.) $4x + 7y + 5z = 51$
(5.) $x + 3y = 14.$ Subtracting (4) from (1).

(2.) $6x + 8y + 10z = 74.$

(6.) $\underline{45x + 15y + 10z = 290.}$ Multiplying (3) by 5.

(7.) $\underline{39x + 7y \qquad = 216.}$ Subtracting (2) from (6).

We now have two equations with two unknown quantities, and will proceed, by the same method, to eliminate y.

(8.) $7x + 21y = 98 =$ equation (5) × 7.

(9.) $\underline{117x + 21y = 648} =$ equation (7) × 3.

$110x \qquad = 550.$ Subtracting (8) from (9)

$x = 5$, the price of an orange.

Substituting in (5), we have,

$$5 + 3y = 14.$$

$$3y = 14 - 5 = 9.$$

$y = 3$, price of a peach.

Substituting in equation (3),

$$9 \times 5 + 3 \times 3 + 2z = 58.$$

$$2z = 58 - 45 - 9 = 4.$$

$z = 2$, price of a pear.

If we had performed this example by the first method, we should have had from the original equations—

(4.) $$x = \frac{51 - 7y - 5z}{4}.$$

(5.) $$x = \frac{37 - 4y - 5z}{3}.$$ (2) reduced.

(6.) $$x = \frac{58 - 3y - 2z}{9}.$$

Then by (16),

(7.) $$\frac{51 - 7y - 5z}{4} = \frac{37 - 4y - 5z}{3}.$$ And,

(8.) $$\frac{37 - 4y - 5z}{3} = \frac{58 - 3y - 2z}{9}.$$

We have now two equations with two unknown quantities. Clearing of fractions,

(9.) $153 - 21y - 15z = 148 - 16y - 20z.$

(10.) $111 - 12y - 15z = 58 - 3y - 2z.$

(11.) $-21y - 15z + 16y + 20z = 148 - 153.$ Transposing.

(12.) $-12y - 15z + 3y + 2z = 58 - 111.$

(13.) $-5y + 5z = -5.$

(14.) $-9y - 13z = -53.$

(15.) $y - z = 1.$ Changing all the signs in (13), and dividing by 5,

(16.) $y = 1 + z.$

(17.) $y = \dfrac{53 - 13z}{9},$ from equation (14).

(18.) $1 + z = \dfrac{53 - 13z}{9}.$

$$9 + 9z = 53 - 13z.$$

$$9z + 13z = 53 - 9.$$

$$22z = 44.$$

$$z = 2.$$

$$y = 1 + z = 3.$$

$$x = \frac{37 - 4y - 5z}{3} = \frac{37 - 12 - 10}{3} = \frac{15}{3} = 5.$$

It will be seen that this method is more complicated than the third, since it gives rise to fractions. After finding the value of *z*, it is easier to find the values of *x* and *y* by this than by the other method. We should have shortened the process, by substituting the value of *y* in (16), at once in (14); and if we had eliminated *z* instead of *x*, the fractions would have been simpler. We will now perform the same example by the second method.

Resuming the original equations,

(1.) $4x + 7y + 5z = 51.$

(4.) $3x + 4y + 5z = 37$ $= (2)$, divided by 2.

(3.) $9x + 3y + 2z = 58.$

(5.) $z = \frac{37 - 3x - 4y}{5}$, from equation (4).

Substituting in (1), we have,

(6.) $4x + 7y + 37 - 3x - 4y = 51.$

Substituting in (3), we have,

(7.) $9x + 3y + \frac{74 - 6x - 8y}{5} = 58.$

(8.) $x + 3y = 14.$ Reducing in (6),

(9.) $45x + 15y + 74 - 6x - 8y = 290.$ Clearing (7) of fractions.

(10.) $39x + 7y = 216.$ Reducing in (9).

(12.) $x = 14 - 3y.$ Substitute in equation (10),

(13.) $546 - 117y + 7y = 216.$

$$-110y = -546 + 216 = -330.$$

$$+110y = +330.$$

$$y = 3.$$

$$x = 14 - 3y = 14 - 9 = 5.$$

$$z = \frac{37 - 3x - 4y}{5} = \frac{37 - 15 - 12}{5} = \frac{10}{5} = 2.$$

2. A fruiterer sold to A 5 oranges, 6 peaches, and 7 pears, for 75 cents; to B 8 oranges, 9 peaches, and 5 apples, for 94 cents; to C 2 oranges, 8 pears, and 10 apples, for 56 cents; and to D 3 peaches, 6 pears, and 9 apples, for 48 cents. What was the price of each?

Let v = the price of an orange,
x = the price of a peach,
y = the price of a pear,
and z = the price of an apple.

(1.) $5v + 6x + 7y = 75.$

(2.) $8v + 9x + 5z = 94.$

(3.) $2v + 8y + 10z = 56.$

(4.) $3x + 6y + 9z = 48.$

We have here 4 equations with 4 unknown quantities, but it will be observed, that only 3 of the unknown quantities enter into each equation. On examining the equations, we find, that it will be simpler to eliminate x, since the coefficient of x in equation (4), is a divisor of the coefficients of x in equation (1) and (2).

$$\begin{array}{lrl}
(5.) & 6x + 12y + 18z = 96. & \text{Equation (4)} \times 2. \\
(1.) & 5v + 6x + 7y \qquad = 75. & \\ \hline
(6.) & -5v \qquad + 5y + 18z = 21. & \text{Subtracting.} \\ \hline
(7.) & 9x + 18y + 27z = 144. & \text{Equation (4)} \times 3. \\
(2.) & 8v + 9x \qquad + 5z = 94. & \\ \hline
(8.) & -8v \qquad + 18y + 22z = 50. &
\end{array}$$

We now compare equations (3), (6), and (8), each containing the unknown quantities v, y, and z, and proceed to eliminate v.

$$\begin{array}{lrl}
(8.) & -8v + 18y + 22z = 50. & \\
(9.) & 8v + 32y + 40z = 224. & \text{Equation (3)} \times 4. \\ \hline
(10.) & 50y + 62z = 274, & \text{adding equations (8) \& (9).} \\ \hline
(11.) & -10v + 10y + 36z = 42. & \text{Equation (6)} \times 2. \\
(12.) & 10v + 40y + 50z = 280. & \text{Equation (3)} \times 5. \\ \hline
(13.) & 50y + 86z = 322. & \text{Adding (11) and (12),} \\
(10.) & 50y + 62z = 274. & \\ \hline
 & 24z = 48. & \text{Subtracting (10) from (13).} \\
 & z = 2. & \text{Substitute this value in (10),}
\end{array}$$

$$50y + 62 \times 2 = 274.$$

$$50y = 274 - 124 = 150.$$

$$y = 3.$$

Dividing equation (4) by 3, $x + 2y + 3z = 16$. Substitute the values of x and y, $x = 16 - 6 - 6 = 16 - 12.$

$$x = 4.$$

Dividing equation (3) by 2, $v = 28 - 4y - 5z$. Substitute the values of y and z, $v = 28 - 12 - 10 = 28 - 22$.

$$v = 6.$$

3. Divide 125 into four such parts, that, if the first be increased by 4, the second diminished by 4, the third multiplied by 4, and the fourth divided by 4, the sum, difference, product, and quotient shall all be equal.

Let x, y, z, and w, represent the parts.

Then (1.) $x + y + z + w = 125.$

(2.) $x + 4 = y - 4.$

(3.) $x + 4 = 4z.$

(4.) $x + 4 = \frac{w}{4}.$

Now we may obtain from equations (2), (3), and (4), a value for each of the unknown quantities, containing x alone. If then, we substitute these values, respectively, in equation (1), we shall have one equation with one unknown quantity.

(5.) $x + 8 = y.$

(6.) $\frac{x + 4}{4} = z.$

(7.) $4x + 16 = w.$

Substituting, we have,

(8.) $x + (x + 8) + (\frac{x + 4}{4}) + (4x + 16) = 125.$ Reducing,

$$6x + \frac{x + 4}{4} = 101.$$

$$24x + x + 4 = 404.$$

$$25x = 400.$$

$$x = 16.$$

$$y = x + 8 = 24.$$

$$z = \frac{x + 4}{4} = 5.$$

$$w = 4x + 16 = 80.$$

Perform two of the following examples by the *first*, two by the *second*, and two by the *third* methods of elimination. The rest may be done by that method, which seems preferable.

4. A miller sold to one man 12 bushels of wheat, 10 of rye, and 16 of barley, for £9 2*s.*; to another, 7 bushels of wheat, 20 of rye, and 10 of barley, for £8 19*s.*; and to a third, 15 bushels of wheat, 8 of rye, and 20 of barley, for £10 5*s.* What was the price of each per bushel?

5. Find 3 such numbers, that the first with $\frac{1}{2}$ of the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, shall each be equal to 34.

6. If A and B can perform a piece of work in 8 days, A and C in 9 days, and B and C in 10 days, in how many days can each alone perform the same work?

Let $x =$ the days in which A can do it,
$y =$ the days in which B can do it,
$z =$ the days in which C can do it.

$\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, will equal what each can do in one day.

Now A and B do $\frac{1}{8}$ in one day,

A and C $\frac{1}{9}$,

and B and C $\frac{1}{10}$. Hence,

$$(1.) \quad \frac{1}{x}+\frac{1}{y}=\frac{1}{8},$$

$$(2.) \quad \frac{1}{x}+\frac{1}{z}=\frac{1}{9},$$

$$(3.) \quad \frac{1}{y}+\frac{1}{z}=\frac{1}{10}.$$

It is easier to eliminate in these equations, without clearing of fractions, by at once adding or subtracting the equations, to find the value of x, y, or z.

7. Says A to B and C, "If each of you will give me 10 cents, my money will be to what you will both have left, as 4 to 5." Says B to A and C, "If each of you will give me 10 cents, my money will be to what you will then have, as 5 to 4." Says C to A and B, "If you will give me 10 cents each, I shall have twice as much money as both of you." How many cents has each?

8. A certain number consists of three digits. The sum of the digits is 10; also the sum of the first and the last digit is $\frac{2}{3}$ of the second digit; and if 198 be subtracted from the number, the digits will be inverted. What is the number?

Let $x =$ 1st digit or hundreds,
$y =$ 2d digit or tens,
$z =$ 3d digit or units.
$100x + 10y + z =$ the number.

9. Find 3 such numbers, that the first with $\frac{1}{2}$ of the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, shall each be equal to 85.

10. Three persons, A, B, and C, talking of their money, A says to B and C, "Give me $\frac{1}{2}$ of your money, and I shall have \$85." B says to A and C, "Give me $\frac{1}{3}$ of your money, and I shall have \$80." C says to A and B, "Give me $\frac{1}{4}$ of your money, and I shall have \$80." What has each?

11. Three young men, A, B, and C, speaking of their money, A says to B and C, "If each of you will give me \$5, I shall have just half as much as both of you will have left." B says to A and C, "If each of you will give me \$5, I shall have just as much as both of you will have left." C says to A and B, "If each of you will give me \$5, I shall have twice as much as both of you will have left." How much has each?

12. A man, with his wife and son, talking of their ages,

said, that his age, added to that of his son, was 16 years more than that of his wife; the wife said that her age added to that of her son, made 8 years more than that of her husband; and that all their ages together amounted to 88 years. What was the age of each?

13. Three boys, A, B, and C, were playing marbles. First, A loses to B and C as many as each of them has. Next, B loses to A and C as many as each of them now has. Lastly, C loses to A and B as many as each of them now has. After all, each of them has 16 marbles. How many had each at first?

14. A grocer has four kinds of tea, marked A, B, C, and D. When he mixes together 7 pounds of A, 5 of B, and 8 of C, the mixture is worth $1.21 a pound. When he mixes together 3 pounds of A, 10 of C, and 5 of D, the mixture is worth $1.50 a pound. At one time he sold 8 pounds of A, 10 of B, 10 of C, and 7 of D, for $48; and, at another time, he sold 18 pounds of A and 15 of D, for $48. What was a pound of each worth?

15. There are two fractions having the same denominator. Now, if 1 be subtracted from the numerator of the smaller, its value will be $\frac{1}{3}$ of the larger fraction; but if 1 be subtracted from the numerator of the larger, its value will be double that of the smaller. And if the smaller fraction be subtracted from the larger, the value of the resulting fraction will be $\frac{1}{3}$. What are the fractions?

16. A boy bought, at one time, 5 apples, 6 pears, and 4 peaches, for 44 cents; at another time, 7 pears, 5 peaches, and 3 oranges, for 56 cents; at another, 8 apples, 12 peaches, and 5 oranges, for 89 cents; and at another, 10 apples, 3 pears, and 9 oranges, for 74 cents. What did he pay for each kind of fruit?

CHAPTER VI.

POWERS AND ROOTS.

SECTION I.

POWERS OF MONOMIALS.

157. *The product arising from multiplying any factor by itself, is called a* POWER *of that factor.*

Thus, 3×3 or 3^2 (42), is the second power, or *square* of 3; $5 \times 5 \times 5$ or 5^3, is the third power, or *cube* of 5; $a \times a \times a \times a$ or a^4, is the 4th power of a; and $a^2b \times a^2b \times a^2b$, or $(a^2b)^3 = a^6b^3$, is the 3d power of a^2b.

158. *The factor, repeated, is called the* ROOT *of the given power.*

Thus, a is the fifth root of a^5; 3 is the 4th root of 81 or 3^4; $5m^2$ is the 3d root of $(5m^2)^3$.

159. *The* EXPONENT *shows how many times a quantity is to be used as a factor.*

NOTE.—When the exponent is 1, it is usually omitted; thus, $a = a^1$, $m = m^1$, &c.

To find the required power of any given quantity;—

160. *Perform the multiplication indicated.* See (73).

Thus, $(5a^2)^3 = 5a^2 \times 5a^2 \times 5a^2 = 125a^6 = 5^3a^{2 \times 3}$; $(m^3)^4 = m^3 \times m^3 \times m^3 \times m^3 = m^{3 \times 4}$ or m^{12}, $(3m^3 n^2)^2 = 3m^3 n^2 \times 3m^3n^2 = 3^2m^{3 \times 2}n^{2 \times 2} = 9m^6n^4$. Hence, to raise a monomial to any power,

161. *Raise the coefficient to the required power, and each letter to the same p[illegible]er, by multiplying its exponent by that of the power required.*

Thus, $(3x^3y^4)^4 = 3^4x^{12}y^{16} = 81x^{12}y^{16}$.

Since $15 = 5 \times 3$, it will be found that $15^2 = 5^2 \times 3^2$ or $225 = 25 \times 9$. Hence, in a *composite quantity;*

162. *The power of the product of two or more factors, is equal to the product of the same power of each of the factors.*

Thus, $(2 \times 5 \times 7)^3 = 2^3 \times 5^3 \times 7^3$ or $(170)^3$.

If the root is positive, as $+ a$, all its powers will be positive, since + multiplied by + will always give +.

If the root is negative, as $- a$; then,

The 2d power, or $(-a)^2$, will be positive; for $-a \times -a$, (80) $= + a^2$.

The 3d power, or $(-a)^3$, will be negative; for $+ a^2 \times -a = - a^3$.

The 4th power, or $(-a)^4$, will be positive, for $- a^3 \times - a = + a^4$; $(-a)^5 = + a^4 \times - a = - a^5$; and $(-a)^6 = - a^5 \times - a = + a^6$.

It will be seen that all the *odd* powers, as the 1st, 3d, 5th, &c., will be *negative;* while all the *even* powers, as the 2d, 4th, 6th, &c., will be *positive.* Hence,

163. *The* ODD *powers of a negative root, are negative; and the* EVEN *powers are positive.*

To find the 3d power of $\frac{2}{5}$, we shall have $\frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$, or $\frac{2^3}{5^3}$, $= \frac{8}{125}$. Hence, to raise a *fraction* to any power,

164. *Raise both the numerator and the denominator to the required power.*

1. Raise xy to the 4th power.

2. Raise a^2bc to the 6th power.

3. What is the square of $4ab$?

4. Find the mth power of $2cd^2$. *Ans.* $2^mc^md^{2m}$.

5. Find the 9th power of abx^2.

6. Find the 3d power of $-5a^2x$.
7. Find the 4th power of $-7a^2b^3c$.
8. Find the 5th power of $6xyz^3$.
9. Find the 4th power of $-2ab$.
10. Find the 5th power of $-3a^2m$.
11. Find the 6th power of $-4m^2n$.
12. Find the *n*th power of $9x^2y^3$.
13. Find the *m*th power of a^mb^n.
14. Find the 3d power of $-\frac{3x^2y}{2a^3}$.
15. Find the 4th power of $\frac{4}{2x^2y}$.
16. What is the square of $\frac{3a^2b^2}{xy^2}$?
17. Find the 3d power of $\frac{4a^2b^2x}{3d^2}$.

To divide one power of a quantity, as a^5, by another power of the same quantity, as a^3, we subtract the exponent of the divisor from that of the dividend (86. (1)), thus, $\frac{a^5}{a^3} = a^{5-3} = a^2$.

In such examples, three cases may occur:

(1.) The exponent of the divisor may be *less* than that of the dividend.

(2.) It may be *equal* to that of the dividend.

(3.) It may be *greater* than that of the dividend.

Thus, a^7 may be divided by a^4, a^7, or a^9. Let us consider each of these cases.

(1.) $a^7 \div a^4$ or $\frac{a^7}{a^4} = a^{7-4} = a^3$.

(2.) $a^7 \div a^7$ or $\frac{a^7}{a^7} = a^{7-7} = a^0$.

(3.) $a^7 \div a^9$ or $\frac{a^7}{a^9} = a^{7-9} = a^{-2}$.

To find an equivalent for a^0, let us divide a by $a = \frac{a}{a} = 1$; subtracting the exponents, we shall have, $a^{1-1} = a^0$. Then, since $\frac{a}{a} = 1$ and $\frac{a}{a} = a^0$ (16), we have $a^0 = 1$. So $\frac{m^3}{m^3} = 1$, and also $m^{3-3} = m^0$; hence $m^0 = 1$.

In the same way it may be shown, that,

165. *Any quantity with* 0 (zero) *for its exponent, is equal to* 1.

Thus $x \times y^0 = x \times 1$ or x.

In the third case, which always gives rise to *negative exponents*, we shall find, by subtracting the exponents, that $\frac{a}{a^4} = a^{1-4} = a^{-3}$; but cancelling the factors, we shall have, $\frac{a}{a^4} = \frac{1}{a^3}$; therefore, $a^{-3} = \frac{1}{a^3}$.

In the same way b^{-5} may be shown to be equal to $\frac{1}{b^5}$; for $\frac{b^2}{b^7} = b^{2-7} = b^{-5}$, and $\frac{b^2}{b^7} = \frac{1}{b^5}$; hence,

166. *Any quantity with a* NEGATIVE EXPONENT, *is equal to unity divided by the same quantity with a positive exponent.*

If we multiply any quantity, as m, by a quantity with a negative exponent, as a^{-2}, it will be equivalent to $m \times \frac{1}{a^2}$, since $a^{-2} = \frac{1}{a^2}$.

SECTION II.

POWERS OF POLYNOMIALS.

To simply indicate a power of a polynomial, we include it in a parenthesis (see Sec. VI. Chap. II.), and then proceed as though it were a monomial. Thus,

The 4th power of $a + b$ is $(a + b)^4$;
The 3d power of $(m + n)^2$ is $(m + n)^6$;
The mth power of $7(5x + y^2)^3$ is $7^m(5x + y^2)^{3m}$;
The 2d power of $(a + b)(c - d^2)^3$ is $(a + b)^2(c - d^2)^6$.

When we have the product of two or more factors to raise to any given power, as in the last two examples, we must raise *each factor* (162) to the required power.

In the following examples, *indicate* the power.

1. The mth power of $x + y$. n
2. The 3d power of $3m^2 - n$. 3
3. The 2d power of $(a + b)(c - d)$. 2
4. The nth power of $x(2a + 3y)$.
5. The 4th power of $5a(m + n)^2$. $(m + n)^8$
6. The 3d power of $8x^2(c - d)^2(a + 5)^3$.
7. The 8th power of $xy(5m^2 - n)(7x + 3y)^3$.
8. The 5th power of $\dfrac{2m(a + b)}{(3x + y)(c - d)^2}$. (164.)

To develop these powers, we must *perform* the multiplication indicated. See (81). Thus, $(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$. Multiplying,

$$
\begin{array}{rl}
(a+b)^1 = & a + b \\
& \underline{a + b} \\
& a^2 + ab \\
& \underline{\quad + ab + b^2} \\
(a+b)^2 = & a^2 + 2ab + b^2 \\
& \underline{\qquad\qquad a + b} \\
& a^3 + 2a^2b + ab^2 \\
& \underline{\quad + a^2b + 2ab^2 + b^3} \\
(a+b)^3 = & a^3 + 3a^2b + 3ab^2 + b^3 \\
& \underline{\qquad\qquad\qquad a + b} \\
& a^4 + 3a^3b + 3a^2b^2 + ab^3 \\
& \underline{\quad + a^3b + 3a^2b^2 + 3ab^3 + b^4} \\
(a+b)^4 = & a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{array}
$$

Note.—The 4th power of $(a-b)$ is in every respect the same except in the signs of the terms. To determine these, in all cases, we have only to observe the rule for signs in the multiplication of monomials (80). Thus, a^4 must be +, because a is +; $4a^3b$ must be —, because $4a^3$ is +, and is multiplied by $-b$. So, $6a^2b^2$ must be +, because $6a^2$ is +, and b^2, which comes from $-b \times -b$, is also +. In the same manner the signs of all the others are determined. In general, the terms containing the odd powers of the negative term of the binomial (61, (3)) will be —; all the others will be +.

9. Find the 2d power of $3a - b + 2c$. [illegible]
10. Find the 3d power of $a + 1$. [illegible]
11. Find the 3d power of $2a - x + c^3$. [illegible]
12. $(a + x)^5$.
13. $(x - y)^6$.
14. $(a^2 - 5)^3$.
15. Find the 2d power of $\dfrac{2a-3}{b^2+c}$.

SECTION III.

BINOMIAL THEOREM.

From the examples already given, it will be seen that the process of raising a high power of a polynomial by actual multiplication must be very tedious. This labor may be very much lessened by a method discovered by Sir Isaac Newton, called the *Binomial Theorem*. Though this method applies primarily to *binomials*, it may be extended to any polynomial; since we may, by the parenthesis, convert any polynomial to a binomial. Thus, $a + 2x^2 - c$ may be written $(a + 2x^2) - c$, or $a + (2x^2 - c)$.

Now, writing out the powers of any binomial $a - b$, as obtained by the preceding section, we shall have

$$(a - b)^1 = a - b.$$
$$(a - b)^2 = a^2 - 2ab + b^2.$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$
$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$
$$(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

It will be seen on examination, that the terms, signs, exponents, and coefficients in these powers, follow certain invariable laws.

(1.) The number of *terms* in each power is greater by 1 than the index of the power to which the binomial is to be raised. Thus the 2d power has *three* terms; the 4th power, *five* terms; &c.

(2.) In regard to the *signs*, when the binomial contains a negative term, all terms containing the *odd* powers of that term will be negative, and all terms having the *even* powers will be positive (163). If both terms of the binomial are positive, as $(a + b)$, all the terms of the power will be positive.

(3.) In regard to *exponents*, it will be seen that the first term of the power contains the first or *leading term* of the binomial, with an exponent equal to the index of the required power; and that its exponents *decrease* by 1 in each successive term of the power. Thus, in $(a - b)^5$, we have a^5, a^4, a^3, a^2, a^1, and since $a^0 = 1$ (165), we may regard a^0 as entering into the last term.

The second term of the power contains the *second* term of the binomial, with 1 for its exponent; and its exponents *increase* by unity, in each successive term till the last, in which term the exponent will always be equal to the index of the power. Thus, in $(a - b)^5$, we shall have b^0, b^1, b^2, b^3, b^4, b^5. Writing the terms of $(a - b)^7$ with their proper signs (163) and exponents, we shall have a^7, $-a^6b$, $+a^5b^2$, $-a^4b^3$, $+a^3b^4$, $-a^2b^5$, $+ab^6$, $-b^7$.

It will also be noticed that the *sum* of the exponents in any given term, is always equal to the index of the required power. In the same way, write the terms of

$(m + n)^6$. $(a + x)^9$.
$(x - y)^7$. $(c - d)^{10}$.

(4.) In regard to *coefficients*, the coefficient of the *first* term is always 1.

The coefficient of the *second* term is always equal to the exponent of the power.

Thus, in $(a - b)^5$, we shall have a^5 the first term, and $5a^4b$ the second term of the power.

To find the coefficient of the *third* term, multiply 5, the coefficient of the *second* term, by 4, the index of the leading quantity in the same term, and divide by 2, the number which marks the place of the term. Then $\frac{5 \times 4}{2} = 10$, the coefficient of the 3d term. Or, in general,

167. *To find the* COEFFICIENT *of any term, multiply the* *exponent of the leading quantity of the last term formed, by its coefficient, and then divide the product by the number which marks its place.*

Forming the coefficients for the successive terms of $(a-b)^7$, as above, we shall have

$$a^7 = \text{the 1st term.}$$

$$7a^6b = \text{the 2d term.}$$

$$\frac{7 \times 6}{2} = 21a^5b^2 = \text{the 3d term.}$$

$$\frac{21 \times 5}{3} = 35a^4b^3 = \text{the 4th term.}$$

$$\frac{35 \times 4}{4} = 35a^3b^4 = \text{the 5th term.}$$

$$\frac{35 \times 3}{5} = 21a^2b^5 = \text{the 6th term.}$$

$$\frac{21 \times 2}{6} = 7ab^6 = \text{the 7th term.}$$

$$\frac{7 \times 1}{7} = b^7 = \text{the 8th term.}$$

Hence, $(a-b)^7 = a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$. Also $(a+x)^6 = a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$.

It will not be necessary to find *all* the terms by actual multiplication, for it will be observed that the same coefficients are repeated in an inverse order after passing the middle term. If the number of terms in the power is *odd*, there will be *one* term in the middle greater than any other; as in $(a+x)^6$. If the number of terms is *even*, there will be *two* terms in the middle, having the same coefficients, greater than any other. In general, to find the successive terms in any given power of a binomial,

168. (1.) *Multiply the exponent of the leading quantity in the term last found by its coefficient, and divide the product by* *the number which marks its place, for the* COEFFICIENT *of the required term.*

(2.) *Diminish the* EXPONENT *of the leading term, and increase that of the second term of the binomial by* 1.

(3.) *If the term contains an* ODD *power of a* NEGATIVE *quantity, it will be minus; otherwise, it is plus.*

Develop the following powers by the rule just given:

1. $(x-y)^6$.	6. $(a+b)^9$.
2. $(a+x)^5$.	7. $(c+d)^{10}$.
3. $(m-n)^8$.	8. $(x-y)^{13}$.
4. $(m+n)^4$.	9. $(a+x)^{11}$.
5. $(x+y)^7$.	10. $(m-n)^{12}$.

Suppose it is required to raise $a+3bc$ to the 3d power. Indicating the powers, we shall have

$a^3+3a^2(3bc)+3a(3bc)^2+(3bc)^3=$, multiplying, $a^3+9a^2bc+27ab^2c^2+27b^3c^3$.

So $(2ab-x^2)^4=(2ab)^4-4(2ab)^3(x^2)+6(2ab)^2(x^2)^2-4(2ab)(x^2)^3+(x^2)^4=16a^4b^4-32a^3b^3x^2+24a^2b^2x^4-8abx^6+x^8$.

In the same way develop the following powers:

11. $(4ab-5c^2)^2$.	14. $(m-5n^2)^4$.
12. $(a^2-3b)^2$.	15. $(3y+2)^6$.
13. $(5a^2+b)^3$.	16. $(7x^2y^5-10x^5z^2)^5$.

To raise $x+2y+z$ to the third power, let $m=x+2y$, then $(x+2y)+z=m+z$.

$$(m+z)^3=m^3+3m^2z+3mz^2+z^3.$$

$$m=x+2y.$$

$$m^2=x^2+4xy+4y^2.$$

$m^3=x^3+3x^2(2y)+3x(2y)^2+(2y)^3=x^3+6x^2y+12xy^2+8y^3$.

Substituting these values in $(m+z)^3$, we shall have $(x^3+6x^2y+12xy^2+8y^3)+3z(x^2+4xy+4y^2)+3z^2(x$

$+ 2y) + z^3 = x^3 + 6x^2y + 12xy^2 + 8y^3 + 3x^2z + 12xyz + 12y^2z + 3xz^2 + 6yz^2 + z^3$.

The powers *might* have been indicated, without substituting m, thus, $[(x + 2y) + z]^3 = (x + 2y)^3 + 3(x + 2y)^2z + 3(x + 2y)z^2 + z^3$.

These powers developed, will produce the same result as before, but the first method is less liable to error.

$(a + b - c + d)^2 =$ (70, Rem.) the second power of $(a + b) - (c - d)$. Let $m = a + b$ and $n = c - d$, then $m - n = (a + b) - (c - d)$

$$(m - n)^2 = m^2 - 2mn + n^2.$$

$$m = a + b. \qquad n = c - d.$$

$$m^2 = a^2 + 2ab + b^2. \qquad n^2 = c^2 - 2cd + d^2.$$

Substituting these values in $(m - n)^2$

$(a^2 + 2ab + b^2) - 2(a + b)(c - d) + (c^2 - 2cd + d^2) = (a^2 + 2ab + b^2 - 2ac + 2ad - 2bc + 2bd + c^2 - 2cd + d^2$.

Raise the following polynomials to the power indicated:

17. $(a + b^2 - 7)^2$.
18. $(2x - y + z^2)^3$.
19. $(a + b + c)^4$.
20. $(a + b - 2c)^4$.
21. $(a + b - c + x)^3$.
22. $(a^2 - c - 2d)^5$.

NOTE.—Experience will show *which* terms may best be included in the parenthesis. Thus, in the first example, it is easier to involve $a + (b^2 - 7)$, than $(a + b^2) - 7$.

SECTION IV.

ROOTS OF MONOMIALS.

169. *The* ROOT *of any power, is one of the equal factors used to produce that power.*

The terms *power* and *root* are correlative terms; that is, the one always implies the other, (157, 158).

Thus, 3 is the square root of 9, since 9 is the square of 3; and 5 is the 3d root of 125, since $5^3 = 125$; so a is the 4th root of a^4, since $a \times a \times a \times a = a^4$.

170. *The* RADICAL SIGN $\sqrt{\;}$ *is used to indicate the roots of quantities.*

Thus $\sqrt{4}$ means the square root of 4; $\sqrt{a+b}$, the 2d root of $a+b$.

Other roots are denoted by a small figure placed over the radical, called its *index*. Thus, $\sqrt[3]{27a^3}$ denotes the 3d root of $27a^3$, and $\sqrt[5]{7a^2b^2}$ denotes the 5th root of $7a^2b^2$. The sign alone, always denotes the second root.

The following table will be of use in extracting the roots of numbers.

TABLE OF ROOTS AND POWERS.

Roots.	1	2	3	4	5	6	7	8	9	10
Squares.	1	4	9	16	25	36	49	64	81	100
Cubes.	1	8	27	64	125	216	343	512	729	1000
4th powers.	1	16	81	256	625	1296	2401	4096	6561	10000
5th powers.	1	32	243	1024	3125	7776	16807	32768	59049	100000

To obtain the root of any power, we must find one of the equal factors which entered in to form that power. Since $(4b)^2 = 16b^2$, the second root of $16b^2$ must be $4b$.

Since to find any given power of a monomial (161), we raise the coefficient to the required power, and multiply the exponent of each letter by the index of the power, it follows that in extracting the *root* of a monomial, we must *reverse* this process; that is,

171. *Extract the root of the coefficient, and divide the exponent of each literal factor, by the index of the root.*

Since $\sqrt[3]{8\times 27}=\sqrt[3]{8}\times\sqrt[3]{27}=2\times 3=6$; we see that,

172. *The product of the roots of any number of factors, is equal to the root of the product of the same factors.*

Thus $\sqrt{a^8b^{12}}=\sqrt{a^8}\times\sqrt{b^{12}}=a^4b^6$. This is the reverse of (162).

As the root, multiplied by itself the requisite number of times, must always reproduce the power, it follows (163), that

173. (1.) *The* EVEN *roots of a positive power may be either plus or minus,* but,

(2.) *The* ODD *roots of any power must have the same sign as the power.*

Thus, the 2d root of 9 may be either $+3$ or -3; for $+3\times+3=+9$, and $-3\times-3=+9$. So the 4th root of m^8 may be either $+m^2$ or $-m^2$. The double sign ± "plus or minus," is used to designate such roots. Thus $\sqrt{25}=\pm 5$.

But the 3d root of a^3 *must* be $+a$, since $+a\times+a\times+a=+a^3$, and the 5th root of -32 must be -2, since $(-2)^5=-32$ (163), and so with all odd roots.

If we attempt to extract the *even* root of a negative quantity, as the 2d root of -9, we shall find it impossible; since no factor multiplied by itself will give -9, $(+3)^2=+9$, and $(-3)^2=+9$; we can only obtain -9 by multiplying $+3$ by -3; but these are not the *same* factors repeated. So the 4th root of $-a^4$ cannot be obtained, though we may *indicate* such roots. Hence we conclude that,

174. *The even root of a negative quantity cannot be obtained, and is to be regarded as* IMAGINARY.

Indicate the following roots, before extracting them.

1. Extract the square root of $64a^4b^2$.
2. Extract the cube root of $27a^3b^6x^9$.

3. Extract the 4th root of $81a^8x^4y^{12}$.
4. Extract the 5th root of $-32x^5y^{10}$.
5. Extract the 3d root of $-64a^6x^3y^{12}$.
6. Extract the 4th root of $1296a^4b^8x^{16}$.
7. Find the 4th root of $16a^4b^8$.
8. Find the 5th root of $1024m^5n^{10}$.
9. Find the 2d root of $81a^2x^4$.
10. Find the 4th root of $256a^8b^4c^{12}$.
11. Find the 5th root of $-243x^{10}$.
12. Find the 3d root of $-125m^6n^3$.
13. Find the 2d root of $-9x^4y^2$.

When any terms of the given monomial are not exact powers of the required degree, we may indicate the root by the *fractional exponent.* Thus, the 3d root of $7a^2b^3$ is $7^{\frac{1}{3}}a^{\frac{2}{3}}b$, which is read, "7 3d root, *a* 3d root 2d power, *b*." So the 5th root of $32a^{10}x^4$ is $2a^2x^{\frac{4}{5}}$. The fractional exponent, it will be observed, arises naturally from extracting the root by (171). Hence,

175. *The numerator of the* FRACTIONAL EXPONENT *always denotes the power, and the denominator the root, of the given quantity.*

To indicate the root of the whole expression, as the 3d root of $7a^3b$, we may enclose the given quantity in a parenthesis, and place the fractional exponent above it; thus, $(7a^3b)^{\frac{1}{3}} = \sqrt[3]{7a^3b}$.

We may indicate the root of a fraction either by the fractional exponent or the radical sign. Thus, the square root of $\frac{16a^2}{9b^4}$ may be written $\sqrt{\frac{16a^2}{9b^4}}$, or $\left(\frac{16a^2}{9b^4}\right)^{\frac{1}{2}}$, or $\frac{\sqrt{16a^2}}{\sqrt{9b^4}}$, or $\frac{(16a^2)^{\frac{1}{2}}}{(9b^4)^{\frac{1}{2}}}$, and in each case equals $\frac{4a}{3b^2}$, since this quantity multiplied by itself will reproduce the given power. Hence,

176. *Any required root of a* FRACTION *is extracted by taking the root of the numerator and of the denominator separately.*

Indicate and then extract the roots of the following fractions:

14. The 3d root of $\frac{27a^6}{8b^6c^3}$.

15. The 4th root of $\frac{81a^4}{b^8}$.

16. The 5th root of $-\frac{a^5x^{10}}{243b^{15}}$.

17. The square root of $\frac{16a^4b^2}{25x^2y^4}$.

SECTION V.

SQUARE ROOT OF POLYNOMIALS.

In order to illustrate the method of extracting the square root of a polynomial, we will extract the square root of $a^2 + 2ab + b^2$, which we know to be the square of $a + b$ (83).

As the first term of the power, a^2, contains the *square of the first term of the root,* the square root of a^2, or a, must be the first term of the root. Taking away a^2 from the given power, we shall have left $2ab + b^2$.

Since we know (83) that the second term of the power, or $2ab$, contains *twice the first term of the root by the second,* if we divide by twice the first term, or $2a$, we shall obtain b for the second term of the root. Hence, $a + b$ is the root sought. But we have not as yet paid any regard to b^2. In order to find the second term of the root, from the *whole* remainder $2ab + b^2$, or $(2a + b)b$,—we will regard $2ab + b^2$ as a dividend, and having found b by dividing the first term $2ab$ by $2a$, we will *add* b to $2a$, making $2a + b$ the entire divisor. $2ab + b^2$ divided by $2a + b$, will give b as the quotient, which exhausts all the terms of the given power.

This process may be exhibited thus:

$$\begin{array}{rl|ll} & a^2+2ab+b^2 & \underline{a+b} & \text{Root.} \\ & a^2 & & \\ \hline & 2ab+b^2 & \underline{2a+b} & \text{Divisor.} \\ (2a+b)b= & \underline{2ab+b^2} & & \end{array}$$

We will now proceed to extract the square root of $25m^4 + 70m^2x + 49x^2$. This polynomial corresponds to $a^2 + 2ab + b^2$, and the root is extracted in the same way; thus,

$$\begin{array}{l|ll} 25m^4+70m^2x+49x^2 & \underline{5m^2+7x} & \text{Root.} \\ 25m^4 & & \\ \hline \qquad\quad 70m^2x+49x^2 & \underline{10m^2+7x} & \text{Divisor.} \\ \qquad\quad \underline{70m^2x+49x^2} & & \\ \qquad\qquad\quad 0 & & \end{array}$$

Here we extract the square root of $25m^4$ (which corresponds to a^2), and obtain $5m^2$ (corresponding to a), the first term of the root. We then subtract $(5m^2)^2$ from the given power, and the remainder $70m^2x + 49x^2$ corresponds to $2ab + b^2$.

We then take twice the first term of the root, or $10m^2$, for a divisor. This corresponds to $2a$. Dividing $70m^2x$ by $10m^2$, we have $7x$ for the second term of the root, corresponding to b. We then place $7x$ in the root, and also to the right of the divisor, and multiply the sum $10m^2 + 7x$ by the last term of the root $7x$. The product will be equal to the dividend $70m^2x + 49x^2$.

NOTE.—The given polynomial must always be arranged according to the powers of some letter; for instance, the polynomial above given might have been arranged according to the powers of x; thus, $49x^2 + 70m^2x + 25m^4$, instead of according to the powers of m. But it must not be arranged thus, $25m^4 + 49x^2 + 70m^2x$, since the second term of the power must always contain twice the first term of the root by the second (95, (2)). This method of extracting the root will apply to a root consisting of any number of terms, for we may convert such a root into a binomial by the parenthesis. Thus,

$(a + b + c)^2 = [a + (b + c)]^2 = a^2 + 2a(b + c) + (b + c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 = a^2 + (2a + b)b + [2(a + b) + c]c.$ See example below.

Hence, to extract the square root of any polynomial,

177. (1.) *Arrange the given polynomial according to the powers of some letter.*

(2.) *Extract the square root of the first term, and write it as the first term of the root. Subtract its square from the given polynomial, and bring down the remainder for a dividend.*

(3.) *Double the root already found for a divisor, and divide the first term of the dividend by it. The quotient will be the second term of the root, and must be annexed, with its proper sign, both to the root and to the right of the divisor.*

(4.) *Multiply the divisor thus increased by the second term of the root, and subtract the product from the dividend; the remainder will form a new dividend.*

(5.) *Double the whole root for a new divisor, and proceed as before, until the entire root of the given polynomial is extracted.*

Remark.—No *binomial* can be a second power; for the square of a monomial is always a monomial, and the square of a binomial always consists of *three* terms (83).

Extract the square root of $9x^4 - 12x^3 + 16x^2 - 8x + 4$.

$$
\begin{array}{lll}
9x^4 - 12x^3 + 16x^2 - 8x + 4 & \underline{|\,3x^2 - 2x + 2} & \text{Root.}\\
9x^4 & & \\
\hline
-12x^3 + 16x^2 & \underline{|\,6x^2 - 2x} & \\
-12x^3 + 4x^2 & & \\
\hline
+12x^2 - 8x + 4 & \underline{|\,6x^2 - 4x + 2} & \\
+12x^2 - 8x + 4 & & \\
\hline
0 & &
\end{array}
$$

Extract the square roots of the following quantities:

2. $36a^6m^4 - 60a^3m^2y^2 + 25y^4$.

3. $a^2 + 2ab + b^2 - 2ac - 2bc + c^2$. $a + b - c$

4. $x^4 - 4x^3 + 6x^2 - 4x + 1$. $> x^2 - 2x -$

5. $4x^6 + 12x^5 + 5x^4 - 2x^3 + 7x^2 - 2x + 1$. $2x^3 + 3x^2 -$

6. $1 - 4a + 4a^2 + 2x - 4ax + x^2$. $x - 2a + 1$

7. $x^6 - 2x^5 + 3x^4 - 2x^3 + x^2$. $= x^3 - x^2 + x$

8. $x^4 + 4x^2y + 4y^2 - 4x^2 - 8y + 4$. $x^2 + 2y - 2$

9. $9a^4b^2 - 30a^2b + 12a^4b^4 + 25 - 20a^2b^3 + 4a^4b^6$. $3a^2 -$

10. $4a^2 - 4ab + b^2 + 12ac - 6bc + 9c^2$. $= 2a - b +$

11. $x^6 + 4x^5 + 10x^4 + 20x^3 + 25x^2 + 24x + 16$. $x^3 + 2x$

12. $4a^6 - 4a^4 + 12a^3 + a^2 - 6a + 9$. $2a^3 - a + 3,$

13. $\dfrac{a^2 + 2ab + b^2}{c^2 - 2cd + d^2}$. $= \dfrac{a + b}{c + d}$

14. $\dfrac{16x^2 + 48xy + 36y^2}{9a^2 - 12ac + 4c^2}$. $\dfrac{4x + 6y}{3a - 2c}$

15. $\dfrac{25a^2 - 90a + 81}{36 + 24x + 4x^2}$. $\dfrac{5a - 9}{6 + 2x}$

SECTION VI.

SQUARE ROOT OF NUMBERS.

The method of extracting the square root of numbers is entirely similar to that already explained in the last Section. Attention should, however be paid to the relative value of the figures in the root.

By referring to the table in Section IV., we shall see

178. (1.) *That the square of any number containing* UNITS *alone lies between* 0 *and* 100, *or* 10^2.

(2.) *That it never contains more than two figures.*

(3.) *That the root is derived directly from the given power.*

If we multiply each of the nine unit figures by 10, that is, remove each to the ten's place, and square the result, we shall have

$$10^2 = 100, \quad 40^2 = 1600, \quad 70^2 = 4900,$$
$$20^2 = 400, \quad 50^2 = 2500, \quad 80^2 = 6400,$$
$$30^2 = 900, \quad 60^2 = 3600, \quad 90^2 = 8100.$$

Hence,

a^2b^3

179. (1.) *The square of any number containing* TENS *alone lies between* 100, *and* 10,000, *or* 100^2.

$3x+4$

(2.) *That its last two places are always zeros; hence it has no significant figures below hundreds.*

(3.) *That the root is derived at once from the power as above.*

If the number is composed of *tens* and *units*, as $25 = 20 + 5$, its square follows the same law as that of any binomial. Let a = the tens and b the units; then $a + b = 20 + 5$, and $(a + b)^2$, or $a^2 + 2ab + b^2 = (20 + 5)^2$, or $(20)^2 + (2 \times 20 \times 5) + (5)^2 = 625$. Hence,

180. (1.) *The square of a number containing tens and units is equal to the square of the tens, + twice the product of the tens by the units, + the square of the units.*

(2.) *Though these different parts are blended in the numerical addition, yet we know* (178) *that the square of the units cannot be above the tens, nor the square of the tens* (179) *below the hundreds.*

(3.) *The root is derived from the power, by reversing the steps by which the power is formed.*

Thus, let it be required to extract the square root of 625. We see at once that its root must be composed of tens and units; for, if it were composed of *units* alone, the square would contain but two figures (178, (2)); and if it were composed of *tens* alone, the power would have zeros in the last two places (179, (2)). Then, the root may be represented by $a + b$, and $(a + b)^2$, or $a^2 + 2ab + b^2$ will equal 625.

We will first find a or the tens of the root. The tens are derived from a^2, or the square of the tens (180, (1)). As the square of the tens will be found among the hundreds, we may set aside, or point off the last two figures of the power, and seek the greatest square in 600, which we know (179) is 400. Hence, $400 = a^2$, and $a = 20$, the tens. If from

$$\begin{aligned} a^2 + 2ab + b^2 &= 625 \text{ we take}\\ a^2 \qquad\qquad &= 400,\\ \text{we have left}\qquad 2ab + b^2 &= 225, \text{ or by } (93\ (1)),\\ (2a + b)b &= 225.\end{aligned}$$

To find b, we simply divide $2ab + b^2$ by the factor $2a + b$; and, to find its numerical value, we divide 225 by the numerical equivalent of $2a + b$. A part of this, $2a = 2 \times 20$, is already known; the part b must be found by using $2a = 40$ as a *trial* divisor. The quotient $5 = b$ added to $40 = 2a$, gives us the required factor $40 + 5$, or $45 = 2a + b$. Multiplying by 5, we have 225 the remainder.

We may exhibit this operation thus:

$$\begin{array}{r|ll} a^2 + 2ab + b^2 = \dot{6}2\dot{5} & 20 = a & \textit{Tens.}\\ a^2 \qquad\qquad = 400 & & \\ \hline 2ab + b^2 = 225 & 40 = 2a & \textit{Trial divisor.}\\ (2a + b)b = 225 & \ \ 5 = b & \textit{Units.}\\ \cline{2-2} & 45 = 2a + b & \textit{Complete divisor.}\end{array}$$

When there are *one* or *two* figures in the square, we take the root directly from the table (178, (3)). When there are *three* figures, we first place a point as above, over the unit figure, and over the second figure to the left, so as to separate the hundreds, which contain the square of the tens, from the tens and units, thus making two periods which will represent the number of figures in the root. When there are *four* figures in the square, the left hand period will contain two figures, owing to the greater value of the tens of the root (179).

If the square contain *five* figures, as in 18225, we first point off the tens and units as above, knowing that the square of the tens must lie above these two. But as 182 hundreds contain three figures, we know (180) that there must be two figures in the square of the tens; and since we seek first the highest, we point off the lowest two, namely, 82. It is as though we had the simple question to extract the square root of 182. So if there were *six, seven*, or more figures in the square, we should point off into periods of two figures each, till we come to the highest, which will contain either one or two figures.

Let the learner explain as above the following example:

$$\begin{array}{r|l l}
\dot{9}02\dot{5} & \underline{90+5} & \textit{Root.} \\
8100 & 180 & \textit{Trial divisor.} \\
\hline
 & 5 & \\
925 & \overline{185} & \textit{Complete divisor.} \\
\underline{925} & & \\
0 & &
\end{array}$$

Extract in the same way the square root of the following numbers:

1. 196.
2. 576.
3. 1225.
4. 841.
5. 1024.
6. 9409.

To shorten the work, we may omit the zeros, remembering to place the second figure of the root at the right of the preceding figure. As the divisor $2a$ represents the *tens*, we must try it with the *tens* of the dividend, to obtain b.

The preceding example, performed in this way, will stand thus,

$$\begin{array}{l|l}
\dot{9}02\dot{5} & \underline{\dot{9}\dot{5}\ \text{Root.}} \\
81 & \\
\hline
92'5 & 185 \\
\underline{925} &
\end{array}$$

When the root contains three figures, as 385, the process is the same. That is, we find the first two figures of the root, as though that were the entire root. We then still represent the root by the binomial $a + b$, by regarding it as composed of 38 tens + 5 units. The divisor $2a$, in finding the third root figure would be twice 38 tens. We will now extract the square root of $385^2 = 148225$.

$$\begin{array}{l|l}
\dot{1}48\dot{2}2\dot{5} & \underline{385 \text{ Root.}} \\
\;9 & \\
\overline{58'2} & \underline{68} \\
544 & \\
\overline{\;382'5} & \underline{765} \\
\;\;\underline{3825.} &
\end{array}$$

Here we find the root 38, as though we had simply to extract the square root of 1482; the remainder 3825 represents what is left after subtracting 38^2 from the given power. Now squaring 38 tens + 5 units, we shall have $38^2 + 2 \times 38 \times 5 + 5^2$, remembering always the relative value of the figures. Taking away 38^2, we shall have left twice 38 tens + 5 units, both multiplied by 5 units. We then divide 382 tens by twice 38 tens or 76 tens,—the quotient 5 is the third root figure, and must be placed in the root and at the right of the trial divisor. Then $765 \times 5 = 3825$, the given remainder.

The square root of 43264 is extracted as follows,

$$\begin{array}{l|l}
\dot{4}3\dot{2}6\dot{4} & \underline{208 \text{ Root.}} \\
4 & \\
\overline{\;32} & 4 \\
\overline{\;3264} & \underline{408} \\
\;\underline{3264.} &
\end{array}$$

In this example the divisor 4 is not contained in the tens of the dividend, we therefore add a zero both to the root and to the divisor, and bring down the next period.

We have then the following rule for extracting the second root of numbers.

181. (1.) *Separate the given number into periods of two figures each, beginning at the right. The left hand period may contain one or two figures.*

(2.) *Find the greatest 2d power in the left hand period, and write its root in the place of a quotient, at the right of the power. Subtract its square from the first period, and to the remainder bring down the next period for a dividend.*

(3.) *Double the root already found for a divisor, and see how many times this divisor is contained in the above dividend rejecting the right hand figure. The quotient will be the second root figure, and must be annexed both to the root, and to the divisor.*

(4.) *Multiply the divisor thus increased, by the last root figure; subtract the product from the dividend; and to the remainder bring down the next period for a new dividend.*

(5.) *Double the whole root already found, for a new divisor, and try it in the tens of the last dividend, for the third root figure, and proceed as above until all the periods have been brought down.*

Extract the square root of the following numbers.

7. 2401. 49
8. 4624.
9. 6084.
10. 140625.
11. 1522756.
12. 5499025.
13. 28153636.
14. 27699169.
15. 9247681.
16. 3637175481.

SECTION VII.

SQUARE ROOT OF FRACTIONS, APPROXIMATE ROOTS.

The square root of a fraction (176) is found by extracting the square root of the numerator and of the denominator separately. Thus the square root of $\frac{25}{36}$ is $\frac{5}{6}$.

Mixed numbers should always be reduced to improper fractions, before extracting the root.

Extract the square root of the following fractions.

1. $\frac{9}{16}$.	6. $30\frac{1}{4}$.
2. $3\frac{13}{36}$.	7. $\frac{529}{1225}$.
3. $\frac{64}{81}$.	8. $13\frac{524}{625}$.
4. $5\frac{19}{25}$.	9. $\frac{729}{6889}$.
5. $7\frac{1}{9}$.	10. $\frac{1024}{9801}$.

By referring to the table of roots and powers, it will be seen that there are many numbers, whose square roots cannot be exactly obtained. For instance, we cannot find the exact square root of 5, or 6, or any number between 4 and 9, and so with many other numbers. We can however obtain the square root of such numbers *approximately*, or nearly.

To approximate the root of a fraction, as $\frac{5}{9}$, where the denominator only is a perfect square;—

182. *Take the root of the denominator, and the nearest root of the numerator, and place after the result the sign + or —, according as it is greater or less than the true root.*

Thus, $\sqrt{\frac{5}{9}} = \frac{2}{3} +$, that is, the true root is *more* than $\frac{2}{3}$. So $\sqrt{\frac{8}{16}} = \frac{3}{4} -$, or the root is less than $\frac{3}{4}$.

To approximate the root of a fraction, where *neither* term is an exact power, as $\frac{3}{7}$,

183. *Make the denominator a perfect square by multiplying both terms by itself, and then proceed as above.*

Thus, $$\left(\frac{3}{7}\right)^{\frac{1}{2}}=\left(\frac{21}{49}\right)^{\frac{1}{2}}=\frac{5}{7}-.$$

If it is required to approximate the root of a fraction to any given degree of accuracy, as to find the square root of $\frac{3}{7}$ to within $\frac{1}{21}$,—we must multiply both terms of the given fraction, by such a square, as will make the denominator a square of the required degree. Thus, as $21=7\times 3$, $21^2=7^2\times 3^2$.

Hence, both terms of $\frac{3}{7}$ must be multiplied by 7 and by 3^2. It is only necessary (162) to *indicate* the product of the denominator. Hence,

$$\left(\frac{3}{7}\right)^{\frac{1}{2}}=\left(\frac{3\times 7\times 9}{7\times 7\times 9}\right)^{\frac{1}{2}}=\left(\frac{189}{49\times 9}\right)^{\frac{1}{2}}=\frac{14}{21}-.$$

In the same manner extract the square roots of the following fractions.

11. $3\frac{1}{7}$.
12. $\frac{3}{5}$.
13. $5\frac{4}{11}$,
14. $\frac{7}{8}$.
15. $\frac{3}{5}$ to within $\frac{1}{20}$.
16. $\frac{5}{8}$ to within $\frac{1}{18}$.
17. $\frac{7}{12}$ to within $\frac{1}{48}$.
18. $\frac{9}{11}$ to within $\frac{1}{22}$.

To approximate the square root of any entire quantity,

184. *Change it to a fraction, having any second power for its denominator, and then proceed as above.*

The second powers of 10, are generally employed, and the result is stated in the form of a decimal. Thus $7=\frac{700}{100}$, and

$$\sqrt{\frac{700}{100}}=\frac{26}{10}+\text{ or }2.6+.$$

If we wished to approximate the root to a greater degree of accuracy, we might multiply by 100^2 or 10000 and then extract the root of $\frac{70000}{10000}$, which would be expressed in hundredths. Instead of $\frac{700}{100}$, we generally write 7.00, regarding the two zeros annexed as decimals, and then extract the square root by (181). Thus,

$$\begin{array}{l|l} 7.00 & \underline{2.6+} \\ 4 & \\ \hline 30'0 & \underline{46} \\ 276 & \\ \hline 24 & \end{array}$$

If now we wish to obtain another decimal, in the root, we may multiply again by 10^2, or what will be the same thing, annex two zeros to the power. Bringing down these zeros, and continuing as above, we should have,

$$\begin{array}{l|l} 7.0000 & \underline{2.65-} \\ 4 & \\ \hline 300 & \underline{46} \\ 276 & \\ \hline 240'0 & \underline{525} \\ \underline{2625} & \end{array}$$

2.65 — is nearer the true root than 2.64 +. It will be seen from the preceding remarks, that the square root of any number always contains twice as many decimals as the root; hence, to extract the square root, to within any required degree of accuracy,

185. *Add as many zeros as are necessary to make the number of decimals in the power twice as many as the number required in the root.*

These zeros need not be annexed at once to the power, but may be added in periods of two each to the several remainders as needed.

We will now extract the square root of 7, carrying the root to five decimals.

```
7        |2.64575 +   Root.
4
30'0     |46
27 6
  240'0    |524
  209 6
    3040'0   |5285
    2642 5
      39750'0   |52907
      37034 9
        271510'0   |529145
        264572 5
           69375
```

This root is accurate to one hundred thousandth, and there is no limit to the accuracy with which it may be approximated.

To extract the square root of 851.24651, we shall find, on pointing in periods of two figures each from the unit figure, that the root will contain two places of integers, and three of decimals. It will be necessary to annex a zero to complete the last period in the decimals (185). The root is extracted as below.

```
8̇51̇.24̇65̇10̇   |29.179 +   Root.
4
45'1   |49
44 1
 102'4   |851
   58 1
   4436'5   |5827
   3978 9
     55761'0   |58349
     52514 1
      3247 9
```

In extracting the square roots of the following numbers, carry the decimals to three places.

19. 5.
20. 2.
21. 823.
22. 527.
23. 8.74.
24. 92.5.
25. .968.
26. 246813.579.

CHAPTER VII.

EQUATIONS OF THE SECOND DEGREE.

SECTION I.

PURE EQUATIONS.

An equation of the second degree, or quadratic equation, contains the second power of the unknown quantity (130). When the second power only enters into the equation, it is said to be a *pure* or *incomplete* quadratic.

1. What number is that which being multiplied by itself and the product doubled will give 162?

Let $x =$ the number,

Then $2x^2 = 162$. Dividing by 2,

$x^2 = 81$. Hence,

Extracting the square root of each member, $x = +9$, or -9, since either squared will give 81. (See 173.)

2. A gentleman being asked his son's age, replied, that if the square of his son's age were subtracted from his own age, which was 30 years, and the remainder were multiplied by his son's age, the product would be 5 times his age. How old was his son?

Let $x =$ the son's age,

Then $(30 - x^2)x = 5x$. Divide by x,

$30 - x^2 = 5$. Transpose,

$-x^2 = -25$. Change the signs,

$x^2 = 25$. Extract the square root,

$x = +5$, or -5.

3. Two numbers are to each other as 3 to 4, and the difference of their squares is 112. Required the numbers.

$x =$ larger,

$\frac{3x}{4} =$ the smaller,

Then $x^2 - \frac{9x^2}{16} = 112$. Clearing of fractions,

$16x^2 - 9x^2 = 1792$. Reducing,

$7x^2 = 1792$. Dividing by 7,

$x^2 = 256$. Extracting square root,

$x = +16$, or -16.

$\frac{3x}{4} = 12$.

4. There is a certain room, the sum of whose length and width is to its length as 5 to 3; and the same sum, multiplied by the length, is equal to 960 square feet. What are the dimensions of the room?

Let $x =$ the length of the room,

and $y =$ the width.

(1.) $x + y = \frac{5x}{3}$.

(2.) $(x + y)x = 960$.

(3.) $3x + 3y = 5x$. Transposing and reducing,

(4.) $y = \frac{2x}{3}$. Substitute in equation (2).

(5.) $\left(x + \frac{2x}{3}\right)x = 960$.

$$x^2 + \frac{2x^2}{3} = 960.$$

$5x^2 = 2880.$ Dividing by 5,

$x^2 = 576.$ Extracting the square root,

$x = +24, \text{ or } -24.$

$$y = \frac{2x}{3} = 16, \text{ or } -16.$$

To solve a pure quadratic equation,

186. *Reduce as in equations of the first degree, and then extract the square root of each member.*

5. A boy bought a number of oranges for 36 cents, and the price of an orange was to the number bought as 1 to 4. How many oranges did he buy, and what did he give apiece?

6. A merchant sold a quantity of flour for a certain sum, and at such a rate, that the price of a barrel was to the number of barrels as 4 to 5; if he had received 45 dollars more for the same quantity, the price of a barrel would have been to the number of barrels as 5 to 4. How many barrels did he sell, and at what price?

7. A gentleman exchanges a field, 81 rods long and 64 rods wide, for an equal quantity of land in the form of a square. What was the side of the square?

8. How long and wide is a rectangular field containing 864 rods, the width of which is equal to $\frac{2}{3}$ of the length?

9. A certain street contains 144 rods of land; and if the length of the street be divided by its width, the quotient will be 16. How long and wide is the street?

10. A trader sold two pieces of broadcloth, which together measured 18 yards; and he received as many dollars a yard for each piece as it contained yards. Now, the sums received for the two were to each other as 25 to 16. How many yards were there in each piece?

Let $x =$ the yards in the longer piece,
and $y =$ the yards in the shorter.
Then $x^2 =$ the price of the first,
and $y^2 =$ the price of the second.
Now $x + y = 18$,
and $x^2 = \frac{25y^2}{16}$.

11. A man divided 14 dollars between his son and daughter, in such a manner that the quotient of the daughter's part, divided by the son's, was $\frac{9}{16}$ of the son's part divided by the daughter's. What was the share of each?

12. A house contains two square rooms, the areas of which are to each other in the proportion of 25 to 9; and a side of the larger room exceeds a side of the smaller by 10 feet. What are the dimensions of the rooms?

13. In a certain orchard there are 4 more rows of trees than there are trees in a row; and if the same number of trees were so arranged that there should be 64 added to each row, the number of the rows would be reduced to 4. How many trees are there in the orchard?

14. When an army was formed in solid column, there were 9 more men in file than in rank; but when it was formed in 9 lines, each rank was increased by 900 men. Of how many men did the army consist?

15. A gentleman has two squares of shrubbery in his grounds, the difference of whose sides is to the side of the greater square as 2 to 9; and the difference of their areas is 128 yards. What are the sides of the squares?

16. Says A to B, "Our ages are the same; but if I were 5 years older, and you were 5 years younger, the product of our ages would be 96." What are their ages?

17. What number is that, which being added to 20 and

subtracted from 10, the product of the sum, multiplied by the difference, will be 81?

18. There is a rectangular field, whose length is to its breadth in the proportion of 6 to 5. A part of this, equal to $\frac{1}{6}$ of the whole, being an orchard, there remained for tillage 625 square rods. What are the length and breadth of the field?

19. It requires 108 square feet of carpeting to cover a certain entry; and the sum of its length and breadth is equal to twice their difference. How long and wide is it?

20. A charitable person distributed a certain sum among some poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received $\frac{1}{3}$ as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. The men received, altogether, 18 shillings more than the women. How many were there of each?

21. A gentleman, being asked the ages of his two sons, replied, that they were to each other as 3 to 4; and that the product of their ages was 48. What were their ages?

22. A gentleman has an oblong garden, of such dimensions that if the difference of the sides be multiplied by the greater side, the product will be 40 square rods; but if the difference be multiplied by the shorter side, the product will be 15 rods. What are the length and width of the garden?

SECTION II.

AFFECTED EQUATIONS OF THE SECOND DEGREE.

It has been stated that a pure or incomplete equation of the second degree contains only the second power of the unknown quantity; as, $x^2 = 16$. An *affected* or *complete* equation of the second degree, is one that contains not only the second, but the first power of the unknown quantity; as, $x^2 + 6x = 135$. Such equations are also called affected quadratic equations.

To solve such equations as $x^2 + 6x = 135$, let us compare the first member with the 2d power of some binomial, as $x + a$; the square of which is $x^2 + 2ax + a^2$.

Writing the corresponding terms under each other, we have,

$$x^2 + 2ax + a^2.$$
$$x^2 + 6x.$$

We see, then, that in both expressions, we have,

x^2 or the square of the first term of the root,

$2ax = 6x$, or twice the first term, multiplied by the second.

Hence, it only remains to add a^2, or the square of the second term of the root, to complete the square.

The square root of x^2 or x is the first term of the root; and since $6x =$ twice the first term of the root, or x, multiplied by the second, if we divide $6x$ by twice the first term, or $2x$, we shall obtain the second term of the root, which is $+ 3$. Adding 3^2 or 9 to $x^2 + 6x$, we have $x^2 + 6x + 9$, a perfect square, whose root is $x + 3$.

Resuming the equation $x^2 + 6x = 135$, we have only to add 9 to the first member to complete the square. Adding the same to the second, to preserve the equality, we have,

$$x^2 + 6x + 9 = 135 + 9 = 144.$$

Extracting the square root of both members, we shall have the binomial $x + 3$ for the square root of the first member, and ± 12 (173), for the square root of the second. Then,

$$x + 3 = \pm 12.$$
$$x = -3 + 12 \text{ or } 9.$$
$$\text{and } x = -3 - 12 \text{ or } -15.$$

Here we have obtained two values of x, one positive and the other negative. Either value will satisfy the equation, but usually the positive value only will answer the conditions of the question. If we substitute both values of x successively in the original equation $x^2 + 6x = 135$, we shall have,

$$(9)^2 + 6 \times 9 \text{ or } 81 + 54 = 135.$$
$$(-15)^2 + 6(-15) \text{ or } 225 - 90 = 135.$$

1. What number is that, whose square diminished by 8 times the number, is equal to 84?

$$\text{Let } x = \text{the number,}$$
$$\text{Then } x^2 - 8x = 84.$$

The first member, $x^2 - 8x$ corresponds to the first two terms of the square of the binomial $x - a$, thus,

$$x^2 - 2ax + a^2,$$
$$x^2 - 8x.$$

Extracting the square root of x^2, we have x for the first term of the root. Dividing the second term of the power $-8x$ by twice the first term of the root, or $2x$, we shall have -4 for the second term of the root, which therefore must be $x - 4$. To complete the square we must add a^2 or $(-4)^2$ to the terms already given. Thus,

$$(x - 4)^2 = x^2 - 8x + 16.$$

As 16 is added to the first member, it must also be added to the second, to preserve the equality. Then,

$x^2 - 8x + 16 = 84 + 16$ or 100. Extracting the root.

$$x - 4 = \pm 10.$$
$$x = 4 + 10 \text{ or } 14.$$
$$x = 4 - 10 \text{ or } -6.$$

2. Given the following equation, to find the value of x.

$87 + 7x^2 - 123 + 3x = 5x^2 + 118 - 5x$. Transposing,
$7x^2 + 3x - 5x^2 + 5x = 118 + 123 - 87$. Reducing,

$$2x^2 + 8x = 154.$$

We must now divide the equation by 2, in order to remove the coefficient from x^2, that it may correspond with the first term of the square of the binomial $x + a$.

Dividing by 2,

$x^2 + 4x = 77$. Completing the square as above,

$x^2 + 4x + 4 = 77 + 4$ or 81. Extracting the sq. root,

$$x + 2 = \pm 9.$$

$$x = -2 + 9 \text{ or } 7.$$

$$x = -2 - 9 \text{ or } -11.$$

3. Divide 34 into two such numbers, that $\frac{1}{5}$ of their product shall be 45.

Let $x =$ one number,

and $34 - x =$ the other.

Then $\frac{(34 - x)x}{5} = 45$. Clearing of fractions,

$$34x - x^2 = 225 \text{ or,}$$

$$-x^2 + 34x = 225.$$

In order to make the first term correspond to the first term of the square of the binomial $x + a$, we must change the sign of each term in the equation, since it is impossible to obtain the square root of $-x^2$ (174.)

Changing the signs,

$x^2 - 34x = -225$. Completing the square,

$x^2 - 34x + 289 = -225 + 289$ or 64. Taking the sq. root,

$$x - 17 = \pm 8.$$

$$x = 17 + 8 = 25.$$

$$x = 17 - 8 = 9.$$

$$34 - x = 9 \text{ or } 25.$$

4. In the following equation, complete the square, and find the value of x.

$3x^2 + 2x = 161$. Dividing by 3,

$$x^2 + \frac{2x}{3} = \frac{161}{3}.$$

Dividing $\frac{2x}{3}$ by $2x$, we have $\frac{1}{3}$ for the second term of the root, the square of which is $\frac{1}{9}$. Then,

$$x^2+\frac{2x}{3}+\frac{1}{9}=\frac{161}{3}+\frac{1}{9}=\frac{484}{9}.$$ Extracting the sq. root,

$$x+\frac{1}{3}=\pm\frac{22}{3}.$$

$$x=7 \text{ or } -\frac{23}{3}.$$

From the preceding examples, it will be seen that all affected equations of the second degree must be reduced to this form, $x^2+2ax=n$, before completing the square. Hence,

187. (1.) *Clear the equation of fractions;—transpose all the terms containing unknown quantities into the first member, and all the terms containing known quantities into the second;—unite the similar terms.*

(2.) *If the term containing x^2 is not positive, make it so, by changing the signs of all the terms in the equation. The remaining terms may be either positive or negative.*

(3.) *If the term containing x^2 has a coefficient, remove it, by dividing the equation by that coefficient.*

Having reduced the equation to the form $x^2+2ax=n$, we complete the square by the following rule:

188. (1.) *To complete the square of the first member, add to it the square of half the coefficient of x. Add the same quantity to the second member, to preserve the equality.*

(2.) *Extract the square root of each member. The square root of the first member will be a binomial, the sign of whose second term will be determined by the sign of the second term of the power. The square root of the second member will have the sign ±.*

(3.) *To find the value of x, transpose and reduce as in equations of the first degree.*

We will now complete the square in the equation

$$x^2 + 2ax = n.$$

$$x^2 + 2ax + a^2 = n + a^2.$$

$$x + a = \pm\sqrt{n + a^2}.$$

$$x = -a \pm \sqrt{n + a^2}.$$

5. What is the value of x in the following equation?

$$ax^2 + c = -bx. \quad \text{Transposing,}$$

$$ax^2 + bx = -c. \quad \text{Dividing by } a,$$

$$x^2 + \frac{bx}{a} = -\frac{c}{a}. \quad \text{Completing the square,}$$

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}, = \frac{-4ac + b^2}{4a^2}$$

Taking the root,

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}, = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

6. Find the value of x in this equation:

$$\frac{x}{x-1} - \frac{x-1}{x} = \frac{9}{20}. \quad \text{Multiply by } 20x,$$

$$\frac{20x^2}{x-1} - 20x + 20 = 9x. \quad \text{Multiply by } x-1,$$

$$20x^2 - 20x^2 + 20x + 20x - 20 = 9x^2 - 9x. \quad \text{Reducing,}$$

$$-9x^2 + 49x = 20. \quad \text{Or changing signs,}$$

$$9x^2 - 49x = -20.$$

$$x^2 - \frac{49x}{9} + \frac{2401}{324} = -\frac{20}{9} + \frac{2401}{324} = \frac{1681}{324}.$$

$$x - \frac{49}{18} = \pm\frac{41}{18}.$$

$$x = \frac{+49 + 41}{18} = \frac{90}{18} = 5.$$

$$x = \frac{+49 - 41}{18} = \frac{4}{9}.$$

In the same way solve the following equations:

7. $x^2 - 14x = 51.$

8. $x^2 + 8x = 105.$

9. $3x^2 + 3x = 216.$

10. $3x^2 - 19x = -6.$

11. $\frac{4x^2}{3} - 11 = \frac{x}{3}.$

12. $\frac{x}{3} + \frac{128}{3} = \frac{x^2}{2} + 20\frac{1}{2}.$

13. $\frac{x}{x+1} + \frac{x+1}{x} = 2\frac{1}{30}.$

14. $\frac{3x}{x+2} - \frac{x-1}{6} = x - 9.$

15. $ax^2 + nx = -b.$

SECTION III.

EXAMPLES PRODUCING AFFECTED EQUATIONS.

1. The ages of a man and his wife amount to 42 years, and the product of their ages is 432. What is the age of each?

2. A gentleman, being asked the ages of his son and daughter, replied, that his son was 5 years older than his daughter, and that the product of their ages was 266. What were their ages?

3. The length of a room exceeds its width by 8 feet, and its area is 768 feet. What are its length and width?

4. The difference of two numbers is 6; and the square of the greater exceeds twice the square of the less by 47. Required the numbers.

5. A gentleman divided 28 dollars between his two sons in such a manner, that the product of their shares was 192. What was the share of each?

6. The wall which encloses a rectangular garden, is 128 yards long, and the area of the garden is 1008 yards. What are its length and breadth?

Let $x =$ the length,
and $64 - x =$ the breadth.

7. There are two numbers whose difference is 9, and $\frac{1}{2}$ their product is 10 more than the square of the smaller number. What are the numbers?

8. The length of a room exceeds its width by 9 feet; and its area is 400 feet. What are the dimensions of the room?

9. It is required to find two numbers, whose sum shall be 14; and such, that 18 times the greater shall be equal to 4 times the square of the less.

10. A man bought a certain number of sheep for 80 dollars. If he had bought 4 more for the same money, they would have come to him a dollar apiece cheaper. How many did he buy?

Let $x =$ the number of sheep.

Then $\frac{80}{x} =$ the price of a sheep,

$$\text{and } \frac{80}{x} = \frac{80}{x+4} + 1.$$

11. In a parcel containing 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins; and each copper coin is worth as many pence as there are silver coins; and the whole is worth 18*s*. How many coins are there of each sort?

12. A drover bought a number of oxen for 675 dollars; which he sold again for 48 dollars a head; and he gained, by

the bargain, as much as he gave for one ox. How many oxen did he buy?

13. Two travellers, A and B, set off at the same time to a place distant 150 miles. A travels 3 miles an hour faster than B, and arrives at his journey's end 8 hours and 20 minutes before him. How many miles did each travel per hour?

Let $x =$ the number of miles B travelled in an hour.

Then $\frac{150}{x} =$ the hours B was on the road,

and $\frac{150}{x+3} =$ the hours A was on the road.

$$\frac{150}{x} - \frac{150}{x+3} = 8\tfrac{1}{3}.$$

14. What two numbers are there, whose sum is 25 and product is 144?

15. The age of A is 12 years more than that of B; and the product of their ages is 640. What is the age of each?

16. The sum of two numbers is 30; and if 18 be added to $\frac{1}{3}$ of their product, the sum will be equal to the square of the smaller number. What are the numbers?

17. A farmer sold a certain number of sheep for £120. If he had sold 8 more for the same money, he would have received 10 shillings less for each sheep. How many did he sell?

18. Two benevolent gentlemen, A and B, distributed each 1200 dollars among a certain number of poor persons. A relieved 40 persons more than B; but B gave 5 dollars more to each person than A. How many persons did each relieve?

19. A person bought two pieces of cloth; the finer of which, at 4 shillings a yard more than the other, cost £18. But the coarser piece, which was 2 yards longer than the finer, cost only £16. How many yards were there in each piece? and what was the price of a yard of each?

20. An officer would arrange 1200 men in a solid body, so that each rank may exceed each file by 59 men. How many must be placed in rank and file?

21. In an orchard containing 900 trees, the trees are so planted that there are 11 more rows than there are trees in a row. Required the number of rows; also the number of trees in a row.

22. The perimeter of a room is 48 feet; and the area of the floor is equal to 35 times the difference of its length and breadth. What are the dimensions of the room?

23. A drover bought a number of sheep for 190 dollars. Having lost 8 of them, he must sell the remainder at a profit of 8 shillings a piece, not to lose money by the bargain. How many sheep did he buy? and at what price?

24. A merchant sold a quantity of sugar for £56, by which he gained as much per cent. as the whole cost him. How much did it cost?

Let $x =$ the cost of the sugar.

Then $56 - x =$ the gain.

If he had gained 6 per cent. of the cost, his gain would have been $\frac{6}{100}$ of x, or $\frac{6x}{100}$. As he gained x per cent, that is $\frac{x}{100}$ of the cost, his gain will be $\frac{x}{100}$ of x. Then,

$$\frac{x^2}{100} = \text{his gain.}$$

$$\text{Therefore, } \frac{x^2}{100} = 56 - x.$$

25. A trader sold a quantity of flour for 39 dollars, and gained as much per cent. as the flour cost him. What did he give for the flour?

26. A butcher bought a certain number of calves for 200

dollars; and, reserving 15, he sold the rest for 180 dollars, by which he gained 2 shillings a head. How many calves did he buy? and at what price?

27. A grass-plot, 18 yards long and 12 wide, is surrounded by a border of flowers of uniform width. The areas of the grass-plot and border are equal. What is the width of the border?

28. A square court-yard has a gravel walk round it. The side of the court wants 2 yards of being 6 times the breadth of the walk; and the number of square yards in the walk, exceeds the number of yards in the periphery of the court by 164. What is the area of the court?

29. What number exceeds its square root by 42?

Let $x^2 =$ the number.

30. A school-boy, being asked the ages of himself and sister, replied, that he was 6 years older than his sister; and that twice the square of her age was 47 less than the square of his own. What were their ages?

31. A gentleman has two square flower-plots in his grounds, which together contain 2120 yards; and the side of the larger plot exceeds that of the smaller by 12 yards. What are the sides of each?

32. A laborer, having built 105 rods of fence, found that, had he built two rods less a day, he would have been 6 days longer in completing the job. How many rods did he build per day?

33. Says A to B, "The product of our ages is 120; and if I were 3 years younger, and you were two years older, the product of our ages would still be 120." What are their ages?

Let $x =$ A's age, and $y =$ B's.

$$\text{Then } xy = 120.$$

$$\text{and } (x - 3)(y + 2) = 120.$$

$$xy - 3y + 2x - \text{[illegible]} = 120, \text{ or}$$

$$120 - 3y + 2x - 6 = 120.$$

$$\text{Then, } \frac{120}{y} = \frac{3y + 6}{2}.$$

34. There are two numbers, such that twice the square of the less, added to the square of the greater, will be 82; and if 6 times the less be subtracted from 4 times the greater, the difference will be 14. What are the numbers?

35. If the greater of two numbers be added to 3 times the less, the sum will be 32; and twice the square of the less, added to one-fourth of the product of the two numbers, will be 93. What are the numbers?

36. A farmer sold 7 calves and 12 sheep for 50 dollars; and the price received for each was such that 3 more calves were sold for \$10 than sheep for \$6. What was the price of each?

Let $x =$ the price of a calf,

and $y =$ that of a sheep.

$$\text{Then } 7x + 12y = 50.$$

$$\text{Also } \frac{10}{x} = \text{calves sold for \$10},$$

$$\text{and } \frac{6}{y} = \text{sheep sold for \$6}.$$

$$\frac{10}{x} = \frac{6}{y} + 3.$$

$$10y = 6x + 3xy.$$

$$10y = x(6 + 3y).$$

$$x = \frac{10y}{6 + 3y}.$$

$$x = \frac{50 - 12y}{7}.$$

37. A young lady, being asked her age, answered, "If you add the square root of my age to half of my age, the sum will be 12." What was her age?

38. A trader bought s[illegible] barrels of flour for $60. Had he bought 3 more barrels for the same sum, each barrel would have cost him one dollar less. How many barrels did he buy?

39. A man had a field whose length exceeded its breadth by 5 rods. He gave 3 dollars a rod to have it fenced; and the whole number of dollars was equal to the number of square rods in the field. Required the length and breadth of the field.

40. Says A to B, "I have 9 dollars more than you, and if the number of dollars we both have, be multiplied by the number that I have, the product will be 266." How many dollars has each?

41. A man has three children, A, B, and C; A being the oldest, and C the youngest. Now, the difference of A and B's ages exceeds the difference of B and C's by 6 years. The sum of all their ages is 33 years, and the sum of the squares of their ages is 467. Required their ages.

42. A farmer sold 80 bushels of wheat and 100 bushels of rye for £65; and each at such a rate that he sold 60 bushels more of rye for £20 than of wheat for £10. What was the price of each?

SECTION IV.

CUBE ROOT.

Since the cube, or third power of any quantity, is found by using that quantity three times as a factor, it follows (169) that the *cube root*, or *third root* of any quantity, must be one of the three equal factors which form it.

The process of finding the first power from the third is called *extracting the cube root.*

Thus, a is the cube root of a^3, since $a \times a \times a = a^3$, and 5 is the cube root of 125, since $5 \times 5 \times 5 = 125$.

By referring to the Table in Sec. IV., Chap. VI., we see,

189. (1.) *That the cube of any number containing* UNITS *alone lies between* 0 *and* 1000, *or* 10^3; and hence,

(2.) *That it never contains more than three figures.*

(3.) *That the root is derived directly from the given power.*

Again, if we multiply each of the nine digits by 10, which is done by removing each into the place of tens, and then cube the result, we shall have

$$\begin{array}{lll} 10^3 = 1000, & 40^3 = 64000, & 70^3 = 343000, \\ 20^3 = 8000, & 50^3 = 125000, & 80^3 = 512000, \\ 30^3 = 27000, & 60^3 = 216000, & 90^3 = 729000. \end{array}$$

From this table we see—

190. (1.) *That the cube of any number containing* TENS *alone lies between* 1000 *and* 1000000, *or* 100^3.

(2.) *That the last three places are zeros; hence, it has no significant figure below thousands.*

(3.) *That the root is derived, as above, from the table of powers.*

If the number is composed of *tens* and *units*, as $45 = 40 + 5$, its cube, like that of $a + b$, will consist of four parts.

Let a represent the tens, and b the units. Then $(a + b)^3 = (40 + 5)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = 40^3 + 3 \times 40^2 \times 5 + 3 \times 40 \times 5^2 + 5^3 = 91125$. Hence, we see,

191. (1.) *That the cube of any number containing* TENS *and* UNITS *is equal to the cube of the tens, + three times the square of the tens by the units, + three times the tens by the square of the units, + the cube of the units.*

(2.) *That these parts are so blended as to disappear in the* *numerical addition; yet that the cube of the units* (189, (1)) *cannot be above the hundreds, nor the cube of the tens* (190, (1)) *below the thousands.*

(3.) *That the root is derived from the power by reversing the steps by which the power is formed.*

Let it be required to extract the cube root of 79507. Its root must contain tens, otherwise the power would have but three figures. It must also contain units, since the last three places are not zeros. See (190) and (191).

Let $a =$ the tens, and $b =$ the units. Then,

$$a^3 + 3a^2b + 3ab^2 + b^3 = 79507.$$

As in the square root, let us first seek the tens. The third power of the tens (190) must be above the hundreds. Hence, the first three figures, or 507, may be set aside, or pointed off. Looking in the table of cubes above, we find the next greatest cube to 79000 to be 64000. Hence,

$$a^3 = 64000, \text{ and}$$
$$a = 40.$$

If now from $a^3 + 3a^2b + 3ab^2 + b^3 = 79507$, we take

$$a^3 \qquad\qquad\qquad\qquad = 64000, \text{ we have left}$$
$$3a^2b + 3ab^2 + b^3 = 15507, \text{ or by } (95, (1)),$$
$$(3a^2 + 3ab + b^2)b = 15507.$$

To find the units from this remainder, we must divide the first member by $3a^2 + 3ab + b^2$, and its equivalent 15507 by the numerical value of the same. But in constructing a divisor from the numerical value of $3a^2 + 3ab + b^2$, we meet with a difficulty. The value of b must be known before we can substitute it in $3ab + b^2$; yet, as $3a^2$ is known, being equal to $3 \times 40^2 = 4800$; and as we need only the leading figures of the divisor to find the quotient figure, we seek how many times this trial divisor, 4800, is contained in 15507. Having thus found $3 = b$, we proceed to make up the numerical value of the divisor.

$$\begin{aligned}
\text{Thus, } 3a^2 &= 4800\\
3ab &= 360\\
b^2 &= 9\\
3a^2+3ab+b^2 &= \overline{5169}\\
\text{Multiplying by } \quad b &= 3, \text{ we have}\\
3a^2b+3ab^2+b^3 &= 15507.
\end{aligned}$$

This operation may be exhibited thus,—

$$\begin{array}{r|ll}
a^3+3a^2b+3ab^2+b^3=7\dot{9}50\dot{7} & 40=a & \textit{Tens.}\\
a^3 \qquad\qquad\qquad =64000 & 3=b & \textit{Units.}\\
\cline{2-2}
\rule{0pt}{0pt} & 4800=3a^2 & \textit{Trial divisor.}\\
\cline{1-1}
3a^2b+3ab^2+b^3=15507 & 360=3ab & \textit{2d term of div.}\\
3a^2b+3ab^2+b^3=15507 & 9=b^2 & \textit{3d term of div.}\\
\hline
0 \quad\;\; 0 \quad 0 \qquad\quad\;\; 0 & \multicolumn{2}{l}{5169=3a^2+3ab+b^2=}\\
 & & \textit{Complete divisor.}
\end{array}$$

In case there are *one, two,* or *three* figures in the given power, the 3d root is taken directly from the table. When there are *four, five,* or *six* figures, we place a point over the unit figure, and another over the third figure to the left, to separate the thousands which contain the cube of the tens from the figures below; thus showing also by the number of periods how many figures there are in the root. If the given power contain *seven, eight,* or *nine* figures, we proceed in the same manner as above. Pointing off three figures, we know that the cube of the tens must be in the figures to the left; and since there are more than three figures left, it is certain that the root of the tens must contain more than one. Pointing off the lowest three, because the cube of the highest figure is not found in these, we seek the highest figure, as if we had a new number containing tens and units in its root.

Thus, to extract the cube root of 19902511, we point off 511, and seek for the cube of the tens in 19902; but since this contains more than three figures, there must be two figures in the root, and the cube of the highest has no part in 902.

Pointing off these last, we seek the greatest cube in 19. The number thus pointed will stand thus:

$$\dot{1}990\dot{2}51\dot{1}.$$

Each period will give one figure in the root. In extracting the root, we will omit the zeros, but must be careful to give each figure its local value. The work will be thus represented:

$\dot{1}990\dot{2}51\dot{1}$	271	*Root.*
8		
119′02	$12 = 3 \times 2^2$	*Hundreds.* Trial divisor.
	$42 = 3 \times 2 \times 7$	*Tens.*
	$49 = 7^2$	*Units.*
	1669 = Complete divisor.	
	7	
11683	11683	
2195′11	$2187 = 3 \times 27^2$	
	$81 = 3 \times 27 \times 1$	
	$1 = 1^2$	
	219511 = *Complete divisor.*	
	1	
219511	219511	

From the above illustrations, we may derive the following rule for the extraction of the cube root:

192. (1.) *Separate the number into periods of three figures each, by placing a point over the unit figure, and over every third figure to the left. The left hand period may contain one, two, or three figures.*

(2.) *Find the greatest cube root in the first period, and place it at the right of the given power for the first root figure. Subtract its third power from the left hand period, and bring down the next period, which must be annexed to the remainder for a dividend.*

(3.) *Take three times the square of the root already found,* *for a trial divisor, and see how many times it is contained in the hundreds of the dividend, and write the result for the second figure of the root. Add to the trial divisor three times the root already found multiplied by the last root figure, and the square of the last root figure, putting each result one place farther to the right than the preceding number. The sum will be the whole or completed divisor.*

(4.) *Multiply the entire divisor by the last root figure, and subtract the product from the dividend. To the remainder annex the next period for a new dividend.*

(5.) *Divide the hundreds of this new dividend, by three times the square of the whole root found, for the third figure of the root; and after completing the divisor as above, multiply the entire divisor by the last root figure. Repeat this process till all the periods have been brought down.*

Remarks.—(1.) If the trial divisor is not contained in the hundreds of the dividend, place a zero in the root, and two zeros to the right of the trial divisor, for a new divisor; and bring down the next period for a new dividend.

(2.) If the product of the completed divisor by the last root figure should *exceed* the dividend, diminish the last root figure by one, and complete the divisor again.

(3.) If there are decimals in the root, there will be three times as many in the power; for $\left(\frac{1}{10}\right)^3=\frac{1}{1000}$, $\left(\frac{1}{200}\right)^3=\frac{1}{8000000}$; that is, $(.1)^3=.001$, and $(.02)^3=.000008$, &c. Therefore, if there are decimals in the power, point from the units every third place to the right, annexing zeros, if necessary, to complete the last period.

(4.) The cube root of any number may be approximated to any degree of accuracy, by annexing three decimal zeros for each decimal required in the root.

(5.) The cube root of a fraction may be approximated by multiplying both terms by the second power of its denominator. Or better, reduce the fraction to a decimal, and extract the root as above. Thus, $\frac{2}{3}=.666$, &c., the root of which may be approximated to any degree of accuracy by annexing .666 as required.

We will extract the cube root of 347 to 3 decimal places.

$$\begin{array}{r|l}
347. & 7.027 + \quad \text{Root} \\
343 & \\
\hline
40'00 & 147 \\
\hline
4000000 & 14700 \\
 & \quad 420 \\
 & \qquad 4 \\
\cline{2-2}
2948408 & 1474204 \times 2 \\
\hline
1051592000 & 1478412 \\
 & \quad 14742 \\
 & \qquad 49 \\
\cline{2-2}
1035920683 & 147988669 \times 7 \\
\hline
15671317 & \text{Rem.}
\end{array}$$

Extract the cube root of the following numbers. If necessary, carry the root to three decimal places.

1. 140608.
2. 79507.
3. 1124864.
4. 12326391.
5. 131.872229.
6. 242970.624.
7. .523606616.
8. $\frac{512}{729}$.
9. $\frac{125}{343}$.
10. $\frac{300763}{456533000}$.
11. 7.
12. 18.
13. 104.
14. .176.
15. .365.
16. .000027.
17. 1123.45.
18. $33\frac{1}{3}$.
19. $\frac{7}{8}$.
20. $6\frac{3}{4}$.

If we have an equation of the 3d degree (130), or cubic equation, it is solved by extracting the cube root of each member. Thus, if

$$x^3 = 125,$$
$$x = 5.$$

Perform the following examples in cubic equations:

21. $\frac{2x^3}{9} - 6 = 156.$

22. Two numbers are to each other as 2 to 3; and the sum of their third powers is 280.

23. There is a certain chest, containing 320 cubic feet. The depth is to the length as 2 to 5, and the width is to the depth as 2 to 1. What are its dimensions?

24. There are two numbers, the sum of whose third powers is 2240, and the difference of these powers is 1216. Required the numbers.

25. There is a certain field whose breadth is to its length as 2 to 5, which is worth $\frac{1}{4}$ as many dollars a square rod as there are rods in the length of the field. The price of the field is $6400. What are its dimensions?

CHAPTER VIII.

IRRATIONAL QUANTITIES, OR SURDS.

SECTION I.

SIMPLIFICATION OF IRRATIONAL QUANTITIES.

THERE are many quantities whose exact roots cannot be taken. Thus, it is impossible to extract the 2d root of 8, the 3d root of 9, or the 4th root of a^3. Hence,

193. *All algebraic quantities are,*

(1.) RATIONAL, *when their exact roots can be taken;—*

(2.) IRRATIONAL, *or* SURDS, *when their exact roots cannot be taken.*

Thus, $\sqrt{9a^2}$ is a rational quantity, because the exact square root of $9a^2$ is $3a$; but $\sqrt{5}$ and $\sqrt[3]{a^2}$ are irrational quantities, or surds.

194. *Irrational quantities are of the same degree when the indices of the roots are the same; otherwise, they are of different degrees.*

Thus, $\sqrt{m}$ and $\sqrt{5}$ are of the same degree; but $\sqrt{p}$ and $\sqrt[5]{7}$ are of different degrees.

195. *Irrational quantities may be expressed in two ways;—*

(1.) *By the* RADICAL SIGN, *in which case they are called radical quantities.*

(2.) *By the* FRACTIONAL EXPONENT, *in which case they are called quantities with fractional exponents.*

Thus, $\sqrt{a} = a^{\frac{1}{2}}$, see (170); $\sqrt[5]{a^3} = a^{\frac{3}{5}}$; $\sqrt[7]{25} = 5^{\frac{2}{7}}$.

In case of numbers, the roots of an irrational quantity may be approximated; but in case of algebraic quantities, they can only be indicated.

Thus, $\sqrt{2} = 1.4142$ nearly; but $\sqrt{m} = m^{\frac{1}{2}}$ cannot be approximated.

Indicate the following roots, both by the fractional exponent and by the radical sign.

1. 2d root of x.
2. 3d root of 15.
3. mth root of d^2.
4. 7th root of $27a^2b^5$.
5. 3d root of $\frac{a}{b^2}$.
6. 4th root of $7x^5y^7$.
7. 3d root of $20a^4b^2$.
8. 8th root of $4x^3y^7$.
9. 5th root of $25a^2b^3c^4$.
10. 3d root of $\frac{12x}{8m}$.

Since the root of any quantity is equal to the product of the roots of its factors (172), all irrational quantities which contain as a factor an exact power of the required degree can be simplified.

Thus, $\sqrt{128a^5y^3}$ is (172) equal to $\sqrt{64a^4y^2} \times \sqrt{2ay}$. Extracting the root of the first factor, we have $8a^2y\sqrt{2ay}$. In the same manner $(7x^5y^7)^{\frac{1}{4}} = (x^4y^4)^{\frac{1}{4}}(7xy^3)^{\frac{1}{4}} = xy(7xy^3)^{\frac{1}{4}}$.

We have, then, the following rule for simplifying irrational quantities:

196. (1.) *Separate the given quantity into two factors, one of which shall be an exact power of the required degree.*

(2.) *Extract the root of this factor, and multiply it by the indicated root of the other factor.*

REMARK.—If the given expression be a polynomial, each of its terms must be divisible by the power of the required degree (96, (1)). Thus $(56a^4m^2 - 64a^3m^3)^{\frac{1}{3}} = (8a^3)^{\frac{1}{3}}\,(7am^2 - 8m^3)^{\frac{1}{3}} = 2a(7m^2 - 8m3)^{\frac{1}{3}}$.

Simplify the following surds:

11. $\sqrt{8m^2n}$.
12. $\sqrt[3]{16a^6x^4}$.
13. $\sqrt{405x^5y^8}$
14. $\sqrt{18m^4n - 36m^2}$.
15. $\sqrt[3]{108a^4b + 27a^3}$.
16. $(50a^3b)^{\frac{1}{2}}$.
17. $(320m^4y)^{\frac{1}{3}}$.
18. $(160x^5y^{10})^{\frac{1}{5}}$.
19. $(162a^4x^3 + 81x^2)^{\frac{1}{2}}$.
20. $(128m^3n^5 - 256m^6n^6)^{\frac{1}{3}}$.

We may reverse this operation, and place any quantity under the radical sign, or within the parenthesis having the fractional exponent, by raising it to a power of the same degree with the given root.

Thus $ab = \sqrt{a^2b^2}$ and $5 = (125)^{\frac{1}{3}}$; $2\sqrt{b} = \sqrt{4} \times \sqrt{b} = \sqrt{4b}$, and $5m(2a)^{\frac{1}{3}} = (125m^3)^{\frac{1}{3}} \times (2a)^{\frac{1}{3}} = (250am^3)^{\frac{1}{3}}$; $\frac{2a}{5}(b)^{\frac{1}{2}} = \left(\frac{4a^2}{25}\right)^{\frac{1}{2}} \times (b)^{\frac{1}{2}} = \left(\frac{4a^2b}{25}\right)^{\frac{1}{2}}$.

In the same way perform the following examples:

21. $am\sqrt{m}$.
22. $\frac{a}{3}\sqrt{5x}$.
23. $2ab\sqrt[3]{6ab}$.
24. $\frac{2}{3}\sqrt[3]{9ax}$.
25. $\frac{m}{5n}\sqrt{2a}$.
26. $2a(b)^{\frac{1}{3}}$.

27. $\frac{1}{4}(12ab)^{\frac{1}{3}}$.

28. $7x^2(2mn)^{\frac{1}{2}}$.

29. $3m^2n^3(3mn)^{\frac{1}{3}}$.

30. $\frac{3a}{4b}(a+b)^{\frac{1}{2}}$.

When the irrational quantity is a *fraction*, it may be simplified by the following rule:

197. *Multiply both terms of the fraction by any quantity that will make the denominator an exact power of the required degree. Separate the resulting fraction into two factors, one of which shall be a perfect power of the required degree. Extract the root of this factor, and indicate that of the other.*

Thus to simplify $\left(\frac{2a}{9b}\right)^{\frac{1}{3}}$, multiply both terms by $3b^2$. We shall then have,

$$\left(\frac{6ab^2}{27b^3}\right)^{\frac{1}{3}} = \left(\frac{1}{27b^3}\right)^{\frac{1}{3}} \times (6ab^2)^{\frac{1}{3}} = \frac{1}{3b}(6ab^2)^{\frac{1}{3}}.$$

To simplify $\sqrt{\frac{4m}{7n}}$, we must multiply by $7n$, thus,

$$\sqrt{\frac{4m}{7n}} = \sqrt{\frac{28mn}{49n^2}} = \sqrt{\frac{4}{49n^2} \times 7mn} = \frac{2}{7n}\sqrt{7mn}.$$

Simplify the following fractional surds :

31. $\left(\frac{3a^2}{5b}\right)^{\frac{1}{2}}$.

32. $\left(\frac{8m}{12n}\right)^{\frac{1}{2}}$.

33. $2\left(\frac{27a^3}{49x}\right)^{\frac{1}{3}}$.

34. $5a\left(\frac{8}{25ab}\right)^{\frac{1}{3}}$.

35. $\sqrt{\frac{m}{6n}}$.

36. $\frac{2}{3}\sqrt{\frac{3a^2}{5b}}$.

37. $\sqrt[3]{\frac{8x}{9y^2}}$.

38. $\frac{2m}{5}\sqrt{\frac{c}{4d}}$

SECTION II.

ADDITION AND SUBTRACTION OF IRRATIONAL QUANTITIES.

Irrational quantities, like other algebraic quantities, are added or subtracted when united with their proper signs.

Thus, $\sqrt{a} + \sqrt{b}$ is the sum of $\sqrt{a}$ and $\sqrt{b}$, and can be reduced to no simpler form.

When expressed by the fractional exponent, the case is the same.

Thus $m^{\frac{1}{3}} + n^{\frac{1}{4}}$, and $x^{\frac{2}{3}} - y^{\frac{1}{5}}$, are irreducible quantities.

A reduction however may take place, when the irrational quantities are similar.

198. *Two irrational quantities are similar when they have precisely the same quantity affected by the same fractional exponents, or are placed under radical signs having indices of the same degree.*

Thus $3\sqrt{a}$ and $5\sqrt{a}$ are similar; so their equals, $3a^{\frac{1}{2}}$ and $5a^{\frac{1}{2}}$; so $\sqrt[6]{x^5}$ and $2\sqrt[6]{x^5}$ or $x^{\frac{5}{6}}$ and $2x^{\frac{5}{6}}$ are similar.

Sometimes irrational quantities apparently dissimilar, become similar by applying the preceding rule for simplification (196).

Thus $\sqrt{50} + 4\sqrt{8}$, are apparently dissimilar; but $\sqrt{50} = \sqrt{2 \times 25} = 5\sqrt{2}$, and $4\sqrt{8} = 4\sqrt{2 \times 4} = 8\sqrt{2}$. Hence,

$$\sqrt{50} + 4\sqrt{8} = 5\sqrt{2} + 8\sqrt{2} = 13\sqrt{2}.$$

In the same manner,

$$12^{\frac{1}{2}} + 75^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 3^{\frac{1}{2}} + 25^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 2 \times 3^{\frac{1}{2}} + 5 \times 3^{\frac{1}{2}} = 7 \times 3^{\frac{1}{2}}.$$

Hence, to add or subtract irrational quantities:

199. *Indicate the addition or subtraction by the proper signs; then simplify if possible, and reduce similar terms.*

NOTE.—The learner should become accustomed to use either the radical sign or the fractional exponent, with equal facility.

Add $a\sqrt{72}$ to $\sqrt{128a^2} - 5\sqrt{2a^2}$.

The indicated sum is $a\sqrt{72}+\sqrt{128a^2}-5\sqrt{2a^2}$. Simplifying, we have $6a\sqrt{2}+8a\sqrt{2}-5a\sqrt{2}=9a\sqrt{2}$.

Subtract $2m(27a)^{\frac{1}{3}}-m(8a)^{\frac{1}{3}}$ from $8(54am^3)^{\frac{1}{3}}$.

Indicating the difference, we have,

$$8(64am^3)^{\frac{1}{3}}-2m(27a)^{\frac{1}{3}}+m(8a)^{\frac{1}{3}}.$$

Simplifying and reducing, we have,

$$32m(a)^{\frac{1}{3}}-6m(a)^{\frac{1}{3}}+2m(a)^{\frac{1}{3}}=28ma^{\frac{1}{3}}.$$

1. Add $a\sqrt{ab}$ and $m\sqrt{ab}$.
2. Add $\frac{1}{2}\sqrt{32}$ and $\sqrt{50}$.
3. Add $m\sqrt[3]{27a^4}$ and $5a\sqrt[3]{am^3}$.
4. Add $\frac{2a}{3}\sqrt[3]{216x}$ and $2\sqrt[3]{a^3x}+a\sqrt[3]{8x}$.
5. Add $\frac{1}{2}\sqrt{\frac{2}{3}}$ and $\frac{1}{4}\sqrt{\frac{1}{6}}$.
6. Add $(243)^{\frac{1}{2}}$ and $5(363)^{\frac{1}{2}}$.
7. Add $2ab^{\frac{1}{2}}$ and $3b^{\frac{1}{2}}m$.
8. Add $5a^{\frac{1}{3}}x$ and $3a^{\frac{1}{3}}x-a^{\frac{1}{3}}x$.
9. Add $\left(\frac{2}{3}\right)^{\frac{1}{3}}$ and $\left(\frac{9}{4}\right)^{\frac{1}{3}}$.
10. Add $2a(50m^3)^{\frac{1}{2}}$ and $b(162m)^{\frac{1}{2}}$.
11. Subtract $\left(\frac{3}{5}\right)^{\frac{1}{2}}$ from $\left(\frac{5}{3}\right)^{\frac{1}{2}}$.
12. Subtract $2a^{\frac{1}{2}}b-3a^{\frac{1}{2}}b$ from $4a^{\frac{1}{2}}b-a^{\frac{1}{2}}b$.
13. Subtract $mx^{\frac{1}{3}}+4mx^{\frac{1}{3}}$ from $5mx^{\frac{1}{3}}+2mx^{\frac{1}{3}}$.
14. Subtract $(250a)^{\frac{1}{3}}$ from $4(16a)^{\frac{1}{3}}$.
15. Subtract $4x(18m)^{\frac{1}{2}}$ from $5(8mx)^{\frac{1}{3}}$.
16. Subtract $\sqrt{16m^3}$ from $\sqrt{25m^3}$.
17. Subtract $\sqrt{\frac{2}{5}}$ from $4\sqrt{\frac{5}{18}}$.
18. Subtract $\sqrt{27}-2\sqrt{12}$ from $\sqrt{363}+5\sqrt{48}$.
19. Subtract $2\sqrt[3]{192a^3}$ from $a\sqrt[3]{24}$.
20. Subtract $2a\sqrt{18}-3\sqrt{2x^2}$ from $m\sqrt{50a^2}$.

SECTION III.

MULTIPLICATION AND DIVISION OF IRRATIONAL QUANTITIES.

To multiply or divide irrational quantities of any degree, we first indicate the operation.

Thus, $\sqrt{a} \times \sqrt{b}$, or $a^{\frac{1}{2}} \times b^{\frac{1}{2}}$, is the indicated product of two irrational quantities of the *same* degree; and $\sqrt{a} \times \sqrt[3]{b}$, or $a^{\frac{1}{2}} \times b^{\frac{1}{3}}$, is the indicated product of two irrational quantities of *different* degrees.

The indicated quotient of the same quantities would be respectively $\frac{\sqrt{a}}{\sqrt{b}} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$, and $\frac{\sqrt{a}}{\sqrt[3]{b}} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{3}}}$.

When the irrational quantities are of the same degree, we may place the product or quotient under the same radical sign, or the same fractional exponent. Thus, in the examples given above, we shall have (172) $\sqrt{ab} = (ab)^{\frac{1}{2}}$ and $\sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{2}}$.

But when the irrational quantities are of different degrees, we cannot take the actual product or quotient as above, until they are reduced to equivalent quantities having a common index.

Thus $\sqrt{a} \times \sqrt[3]{b}$, or $a^{\frac{1}{2}} \times b^{\frac{1}{3}}$, may be reduced to equivalent expressions having roots of the same degree. For $a^{\frac{1}{2}}$, or $\sqrt{a} = a^{\frac{3}{6}}$, or $\sqrt[6]{a^3}$, since if we raise a quantity to any power and then extract the same root, its value will not be changed. Here multiplying the numerator of the fractional exponent by 3, raises the quantity to the 3d power (175), and multiplying the denominator extracts the corresponding root. Hence, $a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}$ and $b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}$. As these quantities are now of the same degree, we may place the product or quotient under the common sign. Then $a^{\frac{1}{2}} \times b^{\frac{1}{3}} = a^{\frac{3}{6}} \times b^{\frac{2}{6}} = (a^3b^2)^{\frac{1}{6}}$; or using the radical $\sqrt{a} \times \sqrt[3]{b} = \sqrt[6]{a^3} \times \sqrt[6]{b^2} = \sqrt[6]{a^3b^2}$, as above. Hence, to reduce irrational quantities to equivalent expressions of the same degree,

200. (1.) *When expressed by a fractional exponent;—*

Reduce the fractional exponents to the least common denominator.

(2.) *When expressed by the radical sign;—*

Multiply each index by that quantity which will make the indices alike, and multiply the exponent of each factor under the radical by the same quantity.

Thus, $a^{\frac{1}{3}} \times 2b^{\frac{3}{4}} = a^{\frac{4}{12}} \times 2b^{\frac{9}{12}} = 2(a^4b^9)^{\frac{1}{12}}$.

So $\sqrt[5]{a^2x} \times 3\sqrt[3]{a^2m} = \sqrt[15]{a^6x^3} \times 3\sqrt[15]{a^{10}m^5} = 3\sqrt[15]{a^{16}m^5x^3}$ $= 3a\sqrt[15]{am^5x^3}$

Hence, to multiply or divide irrational quantities,

201. (1.) *Indicate the operation to be performed.*

(2.) *Take the product or quotient of the coefficients.*

(3.) *If the irrational quantities are of the same degree, place their product or quotient under the same radical sign, or the same fractional exponent.*

(4.) *When the quantities are of different degrees, reduce them to equivalent quantities of the same degree, and then proceed as above.*

NOTE.—The result must always be simplified when possible.

To multiply $\frac{3}{4}\sqrt{\frac{1}{10}}$ by $\frac{5}{6}\sqrt{\frac{4}{5}}$, we shall have $\frac{3}{4}\sqrt{\frac{1}{10}} \times \frac{5}{6}\sqrt{\frac{4}{5}}$ $= \frac{3}{4} \times \frac{5}{6}\sqrt{\frac{1}{10} \times \frac{4}{5}} = \frac{5}{8}\sqrt{\frac{2}{25}} = \frac{5}{8}\sqrt{\frac{1}{25} \times 2} = \frac{1}{8}\sqrt{2}$.

To divide $\frac{7}{8}\sqrt{80}$ by $\frac{3}{4}\sqrt{2}$, we have $\frac{7}{8}\sqrt{80} \div \frac{3}{4}\sqrt{2} = \frac{7}{8}$ $\times \frac{4}{3}\sqrt{\frac{80}{2}} = \frac{7}{6}\sqrt{40} = \frac{7}{3}\sqrt{10}$.

To multiply $a + \sqrt{b}$ by $a - \sqrt{b}$, write the quantities under each other, and multiply as in Sec. VI., Chap. II.

$$\begin{array}{l}
a + \sqrt{b} \\
\underline{a - \sqrt{b}} \\
a^2 + a\sqrt{b} \\
\underline{\quad - a\sqrt{b} - \sqrt{b^2}, \text{ or } b} \\
\quad a^2 - b \quad \text{See (82).}
\end{array}$$

1. Multiply $5a^{\frac{1}{2}}b^{\frac{1}{3}}c$ by $3ab^{\frac{2}{3}}c^{\frac{1}{4}}$.
2. Multiply 4, $3(2)^{\frac{1}{2}}$, and $(2)^{\frac{1}{2}}$.
3. Multiply $6(288)^{\frac{1}{2}}$ by $4(8)^{\frac{1}{2}}$.
4. Multiply $5a^{\frac{1}{3}}b^{\frac{3}{4}}c^{\frac{2}{5}}$ by $3a^{\frac{2}{7}}b^{\frac{1}{2}}c^{\frac{2}{3}}$.
5. Multiply $2a^{\frac{1}{m}}b^{\frac{1}{n}}$ by $4a^{\frac{1}{m}}b^{\frac{1}{n}}$.
6. Multiply $3\sqrt{5}$ by $4\sqrt{5}$.
7. Multiply $7\sqrt{m}$ by $3\sqrt[3]{m^2}$.
8. Multiply $a\sqrt[5]{b}$ by $c\sqrt[3]{d}$.
9. Multiply $7 + \sqrt{3}$ by $7 - \sqrt{3}$.
10. Multiply $a + \sqrt{b}$ by $a - \sqrt{d}$.
11. Multiply $2\sqrt[3]{3}$ by $3\sqrt[3]{9}$.
12. Multiply $\frac{2}{3}\sqrt{\frac{1}{2}}$ by $\frac{3}{4}\sqrt{\frac{4}{18}}$.
13. Divide $6a^{\frac{2}{3}}$ by $2a^{\frac{1}{3}}$.
14. Divide $12b^{\frac{2}{3}}c^{\frac{1}{2}}$ by $3b^{\frac{1}{3}}c^{\frac{1}{4}}$.
15. Divide $15(128)^{\frac{1}{2}}$ by $3(2)^{\frac{1}{2}}$.
16. Divide $\frac{1}{3}(6)^{\frac{1}{2}}$ by $\frac{3}{4}(12)^{\frac{1}{2}}$.
17. Divide $(a + b)^{\frac{5}{4}}$ by $a + b$.
18. Divide $9(ab)^{\frac{1}{2}}$ by $3(b)^{\frac{1}{2}}$.
19. Divide $6\sqrt{a}$ by $2\sqrt[3]{a}$.
20. Divide $5\sqrt[3]{54}$ by $3\sqrt[3]{4}$.
21. Divide $7\sqrt{bc}$ by $14\sqrt{c}$.
22. Divide $a + \sqrt{b}$ by $a - \sqrt{b}$.

Indicating the division, we shall have $\dfrac{a + \sqrt{b}}{a - \sqrt{b}}$. We can make the denominator of the fraction rational, by multiplying both terms by $a + \sqrt{b}$ (82). $\dfrac{(a+\sqrt{b})(a+\sqrt{b})}{(a-\sqrt{b})(a+\sqrt{b})}$ $= \dfrac{a^2 + 2a\sqrt{b} + b}{a^2 - b}$.

23. Divide $10 + \sqrt{3}$ by $10 - \sqrt{3}$.
24. Divide $6 - \sqrt{2}$ by $6 + \sqrt{2}$.
25. Divide $5 + \sqrt{5}$ by $3 - \sqrt{5}$.
26. Divide $m - \sqrt{x}$ by $m + \sqrt{x}$.

SECTION IV.

POWERS AND ROOTS.

To raise an irrational quantity to any given power, we proceed as in the case of any other algebraic quantity, that is, we multiply its exponent by the index of the required power, (161).

Thus the 3d power of $a^{\frac{2}{3}} = a^{\frac{6}{3}}$, and the 6th power of $2(a)^{\frac{1}{2}} = 64a^{\frac{6}{2}}$ or $64a^3$.

If the radical sign is used, the case is the same, for $(\sqrt{m})^6 = \sqrt{m^6} = m^3$, and the 3d power of the $\sqrt[6]{x^5} = \sqrt[6]{x^{15}} = \sqrt{x^5}$ for $(\sqrt[6]{x^5})^3 = (x^{\frac{5}{6}})^3$ or $x^{\frac{15}{6}} = x^{\frac{5}{2}} = \sqrt{x^5}$.

Hence to raise an irrational quantity to any given power,

202. *Multiply the numerator of the fractional exponent, or the exponent of each factor under the radical, by the index of the power.*

1. Find the 2d power of $7a^{\frac{1}{4}}b^{\frac{1}{2}}$.
2. Find the 5th power of $\frac{1}{2}x^{\frac{2}{5}}y^{\frac{1}{3}}$.
3. Find the 3d power of $5a^{\frac{5}{6}}x^{\frac{2}{3}}$.

4. Find the mth power of $a(x-y)^{\frac{2}{3}}$.
5. Find the 3d power of $a^{\frac{1}{n}}b^{\frac{1}{3}}c^{\frac{2}{15}}$.
5. Find the 2d power of $\sqrt{ab}$.
7. Find the 3d power of $\sqrt{xy}$.
8. Find the 4th power of $m\sqrt[6]{(a+b)}$.
9. Find the 3d power of $6\sqrt[9]{(m+n)^2}$.
10. Find the mth power of $a^{\frac{1}{3}}\sqrt{abc}$.

To extract the root of any irrational quantity, we divide the exponent by the index of the required root. (171.)

Thus the 3d root of $a^{\frac{1}{2}} = a^{\frac{1}{6}}$, (125); the 2d root of $9x^{\frac{2}{5}} = 3x^{\frac{1}{5}}$. Using the radical sign we shall have the 3d root of $\sqrt{a} = \sqrt[6]{a}$, and the 2d root of $9\sqrt[5]{x^2} = 3\sqrt[10]{x^2} = 3\sqrt[5]{x}$. So the mth root of $5\sqrt{a+b} = \sqrt[m]{5} \times \sqrt[2m]{a+b}$.

Hence to extract any root of an irrational quantity,

203. *Multiply the denominator of the fractional exponent, or the index of the radical, by the index of the required root.*

11. Extract the 2d root of $49a^{\frac{2}{3}}b^{\frac{1}{3}}$.
12. Extract the 3d root of $3a^{\frac{1}{3}}m^{\frac{6}{7}}$.
13. Extract the 7th root of $xy^{\frac{21}{5}}z^{\frac{3}{4}}$.
14. Extract the mth root of $3a^{\frac{2}{3}}d^{\frac{1}{n}}$.
15. Extract the 3d root of $27a^6(m+n)^{\frac{6}{7}}$.
16. Extract the 2d root of $\sqrt[3]{m^2n^2}$.
17. Extract the 3d root of $8\sqrt{ab}$.
18. Extract the 2d root of $\frac{9}{25}\sqrt[5]{am}$.
19. Extract the mth root of $\sqrt{(a+b)^2}$.
20. Extract the 3d root of $\frac{27}{64}\sqrt[5]{(m-n)^3}$.

SECTION V.

MISCELLANEOUS EQUATIONS OF THE SECOND DEGREE.

1. Given $\sqrt{5+4x}=7$, to find the value of x.

In such examples as the above, where the square root of the first member is put equal to the second, the equation is solved by *squaring both members.* Thus,

$$\sqrt{5+4x}=7.$$
$$5+4x=49.$$
$$4x=44.$$
$$x=11.$$

2. Given, $(25-x^2)^{\frac{1}{2}}+x=7.$ Transposing,

$(25-x^2)^{\frac{1}{2}}=7-x.$ Squaring both members,

$25-x^2=49-14x+x^2.$ Reducing,

$$x^2-7x=-12.$$
$$x^2-7x+\tfrac{49}{4}=\tfrac{1}{4}.$$
$$x-\tfrac{7}{2}=\pm\tfrac{1}{2}.$$
$$x=4 \text{ or } 3.$$

In the same way perform the following examples:

3. $\sqrt{16-x}=3.$
4. $\sqrt{31-2x}+x=2x+2.$
5. $\sqrt{x+11}=\sqrt{x}+1.$
6. $(x-28)^{\frac{1}{2}}=14-(x)^{\frac{1}{2}}.$
7. Given $x+2\sqrt{x}=8.$

This is an affected equation, corresponding to $x^2+2ax=n$. The first term of the root will be $\sqrt{x}$, and the second term will be 1. Whenever an equation, like the above, contains two powers or roots of the unknown quantity, one of which is the square of the other, it may be reduced to the form $x^2 + 2ax = n$, and solved by completing the square.

Thus,
$$x + 2\sqrt{x} + 1 = 9.$$
$$\sqrt{x} + 1 = \pm 3.$$
$$\sqrt{x} = 2 \text{ or } -4.$$
$$x = 4 \text{ or } 16.$$

8. $x - 8\sqrt{x} = 9.$
9. $\sqrt{x} + 6\sqrt[4]{x} = 16.$
10. $x^6 - 6x^3 = 16.$
11. $2x^4 - x^2 = 496.$
12. $5x^{\frac{2}{3}} - x^{\frac{1}{3}} = 18.$
13. Given $x + 2 + 6\sqrt{x + 2} = 27.$

Here, if $\sqrt{x + 2}$ be regarded as the unknown quantity, the equation may be solved by completing the square as above. For convenience in writing, let any letter, as $y = \sqrt{x + 2}$. Substituting in the above equation, we shall have,

$$y^2 + 6y = 27.$$
$$y^2 + 6y + 9 = 36.$$
$$y + 3 = \pm 6.$$
$$y = 3 \text{ or } -9.$$

Now putting $\sqrt{x + 2}$ equal to y, we shall have

$$\sqrt{x + 2} = 3 \text{ or } -9.$$
$$x + 2 = 9 \text{ or } 81.$$
$$x = 7 \text{ or } 79.$$

14. Given $x^2 - 3x - 2\sqrt{x^2 - 3x + 6} = 2.$

In this equation if we add 6 to each member, we shall have the first member reduced to the form $x^2 - 2ax = n$, the unknown quantity being $\sqrt{x^2 - 3x + 6}$. Let $y =$ this quantity; then,

$y^2 - 2y = 8$. By completing the square as above,

$y = 4$ or -2. Substituting,

$\sqrt{x^2 - 3x + 6} = 4$ or -2.

$x^2 - 3x + 6 = 16$ or 4.

$x^2 - 3x = 10$ or -2.

Now completing the square, using both values of the equation, we shall find that x has *four* values; viz. 5 or -2, and 2 or 1.

15. $x + 10 - 3\sqrt{x + 10} = 4.$

16. $x + 5\sqrt{x + 13} = 23.$

17. $x^2 - 2\sqrt{x^2 - 13} = 37.$

18. $2x^2 - 5x + 3\sqrt{2x^2 - 5x + 4} = 24.$

19. $\dfrac{\sqrt{9x} + 6}{\sqrt{x} + 2} = \dfrac{16 - \sqrt{x}}{\sqrt{x}}.$

20. $\dfrac{3\sqrt{x} + 8}{\sqrt{x} + 1} - \dfrac{\sqrt{x} - 1}{\sqrt{x} + 8} = \dfrac{15}{4}.$

21. Given $\begin{cases} (1.)\ 5x - 2y = x + y. \\ (2.)\ x^2 + y^2 = 25. \end{cases}$

$4x = 3y$. Transposing in the 1st,

$x = \dfrac{3y}{4}$. Substituting in the 2d,

$$\frac{9y^2}{16} + y^2 = 25.$$

$$25y^2 = 25 \times 16.$$

$$y^2 = 16.$$

$$y = \pm 4.$$

$$x = \pm 3.$$

22. Given $\begin{cases} (1.)\ xy = 24. \\ (2.)\ x^2 + y^2 = 52. \end{cases}$

Adding twice the 1st to the 2d, and then subtracting twice the 1st from the 2d, we have,

(3.) $x^2 + 2xy + y^2 = 100.$

(4.) $x^2 - 2xy + y^2 = 4.$ Extracting sq. root,

(5.) $x + y = \pm 10.$

(6.) $x - y = \pm 2.$

Adding (5) and (6), $2x = \pm 12$ or $\pm 8.$

Subtracting (6) from (5), $2y = \pm 8$ or $\pm 12.$

$x = \pm 6$ or $\pm 4.$

$y = \pm 4$ or $\pm 6.$

23. Given $\begin{cases} (1.) & x^2 + xy = 84. \\ (2.) & xy + y^2 = 60. \end{cases}$

Adding (2) to (1), $x^2 + 2xy + y^2 = 144.$

Extracting sq. root, $x + y = \pm 12.$

Separating (1) into factors, $x(x + y) = 84.$

$\pm 12\,x = 84.$

$x = \pm 7.$

$y = \pm 5.$

24. Given $\begin{cases} (1.) & x^4 - y^4 = 671. \\ (2.) & x^2 + y^2 = 61. \end{cases}$

(3.) Dividing (1) by (2), $x^2 - y^2 = 11.$

Adding (2) and (3), $2x^2 = 72.$

Subtracting, $2y^2 = 50.$

$x^2 = 36.$

$y^2 = 25.$

$x = \pm 6.$

$y = \pm 5.$

25. Given $\begin{cases} (1.) & x^3 + y^3 = 91. \\ (2.) & x^2y + xy^2 = 84. \end{cases}$

Add three times the 2d to the 1st equation,

(3.) $x^3 + 3x^2y + 3xy^2 + y^3 = 343.$

(4.) Taking the cube root, $x + y = 7.$

(5.) Factoring the 2d, $xy(x + y) = 84.$

(6.) $7xy = 84.$

(7.) $xy = 12.$

We may find the value of x in this equation, and substitute it in the 4th; but it is much easier to square the 4th and subtract 4 times the 7th from it. Thus,

$x^2 + 2xy + y^2 = 49.$ Subtract $4xy = 48,$

$x^2 - 2xy + y^2 = 1.$

$$x + y = \pm 7.$$

$$x - y = \pm 1.$$

Taking the plus value, $x = 4.$

$$y = 3.$$

26. Given $\begin{cases} (1.) & x^2y + xy^2 = 120. \\ (2.) & x^3y^2 + x^2y^3 = 1800. \end{cases}$

(3.) $xy(x + y) = 120.$ Factoring,

(4.) $x^2y^2(x + y) = 1800.$ Divide (4) by (3),

$xy = 15.$ Substituting in 3,

$$x + y = 8.$$

The values of x and y may now be found as above.

27. $\begin{cases} \dfrac{3x - 2y}{x - 1} = 2. \\ 3xy = 120. \end{cases}$

28. $\begin{cases} xy = 63. \\ x^2 + y^2 = 130. \end{cases}$

29. $\begin{cases} x + 4y = 3x. \\ xy = 50. \end{cases}$

30. $\begin{cases} x^2 - xy + y^2 = 109. \\ 2xy = 120. \end{cases}$

31. $\begin{cases} \sqrt{x} + \sqrt{y} = 15. \\ \sqrt{x} - \sqrt{y} = 3. \end{cases}$

32. $\begin{cases} x^6 - y^6 = 728. \\ x^3 - y^3 = 26. \end{cases}$

33. $\begin{cases} x^2 - xy = 28. \\ xy - y^2 = 12. \end{cases}$

34. $\begin{cases} x^2y + xy^2 = 70. \\ x^3y^2 + x^2y^3 = 700. \end{cases}$

35. $\begin{cases} x^2y + xy^2 = 84. \\ x^3y^2 + x^2y^3 = 1008. \end{cases}$

36. $\begin{cases} x + \sqrt{xy} + y = 28. \\ x^2 + xy + y^2 = 336. \end{cases}$

Note. Divide the second equation by the first.

37. $\begin{cases} x^3 + y^3 = 280. \\ x^2y + xy^2 = 240. \end{cases}$

CHAPTER IX.

PROGRESSION AND PROPORTION.

SECTION I.

ARITHMETICAL PROGRESSION.

203. *An* ARITHMETICAL PROGRESSION *is a series of numbers whose terms increase or decrease by the addition or subtraction of the same quantity, called the* COMMON DIFFERENCE.

Thus, 1, 3, 5, 7, 9, 11, is an arithmetical progression, whose common difference is 2.

204. *An arithmetical progression is*—

(1.) INCREASING, *when its successive terms are formed by the addition of the common difference;*

(2.) DECREASING, *when its successive terms are formed by the subtraction of the common difference.*

Thus, 2, 5, 8, 11, 14, is an increasing progression.
13, 11, 9, 7, 5, is a decreasing progression.

To derive a general rule for finding the last term in any arithmetical progression,

Let $a =$ the first term,
$d =$ the common difference,
$n =$ the number of terms,
and $l =$ the last term.

Then the progression, if increasing, will be

$$a,\ (a + d),\ (a + 2d),\ (a + 3d),\ (a + 4d),\ \&c.$$

That is, the 2d term $= a +$ the common difference,
The 3d term $= a +$ twice the common difference,
The 4th term $= a +$ three times the common difference,

and so on, each term being equal to the first term plus the product of the common difference by a number less by 1 than that which marks the place of the terms. The 8th term, for instance, is equal to $a + 7 \times d$, and the nth or last term will be $a + (n - 1)d$. That is, putting l for the nth term,

$$l = a + (n - 1)d.$$

If the progression is decreasing, the formula will be

$$l = a - (n - 1)d.$$

Hence, to find the last term in any progression,

205. *Multiply the common difference by the number of terms less one; add the product to the first term, if the progression is increasing; and subtract it, if the progression is decreasing.*

What is the 20th term of the progression 7, 12, 17, &c.?

Here $a = 7$, $d = 5$, and $n = 20$; then

$$l = 7 + (20 - 1)5 = 7 + 95 = 102.$$

1. What is the 50th term of the series 8, 11, 14, &c.?
2. Find the 20th term of the series 158, 153, 148, &c.
3. Find the 30th term of the progression 2, 4, 6, 8, &c.
4. Find the 12th term of the series 60, 56, 52, &c.
5. The first term in a series by difference is 8, and the common difference is 5. What is the 10th term?

To find a formula for the sum of the series, we will let S represent the sum, which is manifestly obtained by adding the successive terms of the progression. As the value of n is indefinite (52), we cannot write out all the terms, and must therefore use points to supply the place of the terms omitted.

$$\text{Then } S = a + (a + d) + (a + 2d) + \dots\dots + l.$$
$$\text{Also } S = l + (l - d) + (l - 2d) + \dots\dots + a.$$

This last equation is the same series inverted, that is, the last or nth term is written first. The sum will evidently be the same, in whatever order the series is written. Now, if we add these two equations, we shall have,

$$2S = (a + l) + (a + l) + (a + l) + \ldots + (a + l).$$

That is, $2S$ will equal $a + l$ repeated as many times as there are terms, or n times. Then

$$2S = n(a + l).$$

$$S = \frac{n(a + l)}{2}.$$

Hence,

206. *The sum of the terms is equal to half the product of the sum of the first and the last terms by the number of terms.*

What is the sum of 16 terms of the series 5, 9, 13, &c.?

Here $a = 5$, $d = 4$, $n = 16$.

We must first find l or the last term.

$$l = 5 + (16 - 1)4 = 5 + 60 = 65.$$

$$S = \frac{16(65 + 5)}{2} = \frac{16 \times 70}{2} = 8 \times 70 = 560.$$

6. Find the sum of 30 terms of the series 3, 5, 7, &c.

7. What is the sum of 12 terms of the series 100, 95, 90, &c.?

8. Find the sum of 15 terms of the series 50, $49\frac{1}{2}$, 49, $48\frac{1}{2}$, &c.

9. Find the sum of 40 terms of the progression 6, 13, 20, &c.

10. There are 10 flower pots in a straight line 6 feet apart, and a well placed 25 feet from the first flower pot. How far must a boy travel to water the plants, if he return to the well for water after watering each flower?

The above formulas are sufficient for all possible cases; but it is sometimes convenient to find S without first finding l, by substituting the formula for l in that for S. Thus,

$$l = a + (n - 1)d.$$

$$S = \frac{n(a + l)}{2}$$ Hence,

$$S = \frac{n[a + a + (n - 1)d]}{2} = \frac{n[2a + d(n - 1)]}{2}$$

$$= \frac{2an + dn(n - 1)}{2} = an + \frac{dn(n - 1)}{2}.$$

Performing the 6th example by this formula,

$$S = 3 \times 30 + \frac{2 \times 30 \times 29}{2}.$$

$$S = 90 + 30 \times 29 = 960.$$

Perform the 7th, 8th, and 9th in the same way.

The following problems may be solved by the principles given above:

11. Find four numbers in arithmetical progression, whose sum is 78, and whose common difference is 5.

Let $x =$ the first term.

12. Find three numbers in arithmetical progression, such that their sum shall be 63, and the product of the extremes shall be 405.

Let $x =$ the middle term, and y the common difference.

13. Two men, 294 miles apart, set out at the same time to meet each other. The first travels 40 miles the first day, 34 the second, 28 the third, and so on. The second travels regularly 20 miles a day. In how many days will they meet, and how many miles will each travel?

Let $x =$ the number of days, which will also be the number of terms.

14. There is a number consisting of three digits, which are in arithmetical progression. The sum of the digits is 9; and if 396 be subtracted from the number, the digits will be inverted. Required the number.

Let $x =$ the middle digit, and y the common difference.

SECTION II.

RATIO AND PROPORTION.

207. RATIO *is the quotient arising from dividing one quantity by another.*

Thus, $\frac{6}{2}$ or 3 is the ratio of 6 to 2; $\frac{a}{b}$ is the ratio of a to b. Ratio may be expressed either in the form of a fraction, as $\frac{a}{b}$, or by placing two points between the terms; thus, $a : b$; to be read, a is to b.

208. *The two terms of a ratio are called a* COUPLET; *the first is called the* ANTECEDENT, *and the second the* CONSEQUENT.

Thus, $3 : 4$ and $a : b$ are couplets; 3 and a are the antecedents; 4 and b the consequents.

209. *A* PROPORTION *is an equality of two ratios.*

Thus, $3 : 4 = 9 : 12$, or $a : b = c : d$, is a proportion. Instead of the sign $=$, four points $: :$ are generally placed between the equal ratios. Thus, $3 : 4 :: 9 : 12$, which is read 3 is to 4 as 9 is to 12.

210. *In a proportion, the first and last terms are called* EXTREMES; *the middle terms are called* MEANS.

Thus, in the proportion $a : b :: c : d$, a and d are the extremes, b and c are the means.

Since a proportion is composed of two equal ratios, and these may also be expressed in the form of a fraction, it follows that

211. *A proportion may be converted into an equation by putting the fractional forms of the ratios equal to each other.*

Thus, $a : b :: c : d$ is the same as

$$\frac{a}{b} = \frac{c}{d}.$$

Clearing this equation of fractions, we have

$$ad = bc.$$

But ad is the product of the extremes, and bc the product of the means. Hence,

212. *In any proportion the product of the means is equal to the product of the extremes.*

Thus, if $2 : 5 :: 8 : 20$, then $2 \times 20 = 5 \times 8$.

Again, if $ad = bc$, we may divide both members by bd.

Thus, $\frac{ad}{bd} = \frac{bc}{bd}$; or cancelling common factors,

$$\frac{a}{b} = \frac{c}{d}, \text{ or } (209)$$

$$a : b :: c : d. \quad \text{Hence,}$$

213. *If two equal products contain each two factors, the factors of the one may be made the means, and those of the other the extremes of a proportion.*

Thus, $7 \times 8 = 4 \times 14$. Then
$7 : 4 :: 14 : 8$, or
$4 : 7 :: 8 :: 14$.

When three quantities are proportional, as

$a : b :: b : c$, we have (211)

$$\frac{a}{b} = \frac{b}{c}, \text{ or}$$

$ac = b^2$, and $b = \sqrt{ac}$. Hence,

214. *When three quantities are proportional, the square of the middle term is equal to the product of the extremes; and the mean is equal to the square root of the same product.*

Thus, $4 : 8 :: 8 : 16.$

$$8^2 = 4 \times 16.$$

$$8 = \sqrt{4 \times 16} = \sqrt{64} = 8.$$

Note.—When three terms are in proportion, the middle one is said to be a *mean proportional* between the other two.

In the proportion

$$a : b :: c : d, \text{ we have (212)}$$

$$ad = bc.$$

Now we may make any number of changes in the order of the terms of this proportion, provided we still preserve the equation

$$ad = bc.$$

Thus we may invert the means, that is, we may have

$$a : c :: b : d,$$

or we may invert the extremes, thus,

$$d : b :: c : a,$$

or both, as

$$d : c :: b : a.$$

The means and the extremes may exchange places, as

$$b : a :: d : c,$$

and then they may be inverted, as

$$c : a :: d : b.$$

$$c : d :: a : b.$$

$$b : d :: a : c.$$

Hence,

215. *When four quantities are in proportion, they will still be in proportion, whatever changes be made in the order of the terms, provided the product of the means remain equal to the product of the extremes.*

Thus, the following changes may be made in the proportion $2 : 4 :: 8 : 16.$

$2 : 4 :: 8 : 16.$
$2 : 8 :: 4 : 16.$
$4 : 2 :: 16 : 8.$
$4 : 16 :: 2 : 8.$
$8 : 2 :: 16 : 4.$
$8 : 16 :: 2 : 4.$
$16 : 8 :: 4 : 2.$
$16 : 4 :: 8 : 2.$

Remark.—Observe that if the second term is *greater* than the first, the fourth must be greater than the third; and if the second term is *less* than the first, the fourth must be less than the third.

If we have two proportions having a common ratio, as

$$a : b :: c : d,$$
$$a : b :: x : y,$$

we shall have, by (211),

$$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{a}{b} = \frac{x}{y}.$$

Hence (16), $\dfrac{c}{d} = \dfrac{x}{y},$

or (209) $c : d :: x : y.$

Hence,

216 *If two proportions have a common ratio, the remaining ratios will be equal, and form a proportion.*

Thus, if $5 : 15 :: 1 : 3,$
and $1 : 3 :: 7 : 21,$
then $5 : 15 :: 7 : 21.$

EXAMPLES.

1. Make a proportion from $ax = bm$.

2. Find a mean proportional between 6 and 24.

Let $x =$ the mean proportional.

3. Find the value of x in the following proportions (212):

$$x : 5 :: 16 : 20.$$
$$3 : x :: 8 : 50.$$
$$7 : 56 :: x : 112.$$
$$5 : 40 :: 3 : x.$$

4. Make a proportion from $x^2 = mn$.

5\. Given $\begin{cases} x^2 : y^2 :: 100 : 16. \\ 2x - 10 : 2y + 2 :: 14 : 7. \end{cases}$

$$4x = 10y.$$

$$(2x - 10)7 = (2y + 2)14. \quad \text{Divide by 7 and 2,}$$

$$x - 5 = 2y + 2.$$

6\. Given $\begin{cases} x^2 - y^2 : (x - y)^2 :: x + 6 : 3. \\ x : y :: 12 : 8. \end{cases}$

7\. Given $\begin{cases} x : y :: 5 : 8. \\ 2x + y : 2x - 16 :: 9 : 1. \end{cases}$

8\. What third proportion may be derived from the two following proportions?

$$x : 3 :: y : 5.$$
$$x : 9 :: y : 15.$$

SECTION III.

GEOMETRICAL PROGRESSION.

217. *A* GEOMETRICAL PROGRESSION *is a series of terms, each of which is found by multiplying the preceding term by a constant quantity called the common ratio.*

Thus, 2, 6, 18, 54, 162, is an *increasing* geometrical progression, in which the common ratio is 3.

20, 10, 5, $\frac{5}{2}$, $\frac{5}{4}$, is a *decreasing* geometrical progression, in which the common ratio is $\frac{1}{2}$.

The ratio can always be found by dividing any term by the one immediately preceding.

$$\text{Thus, } \frac{162}{54} = 3. \quad \frac{5}{10} = \frac{1}{2}.$$

218. *In a geometrical progression, five things are to be considered;* namely,

(1.) *The first term,* *or* a.
(2.) *The last term,* *or* l.
(3.) *The ratio,* *or* r.
(4.) *The number of terms, or* n.
(5.) *The sum of the terms, or* S.

Thus, in the progression 2, 6, 18, 54, 162, 486, $a=2$, $l=486$, $r=3$, $n=6$, and $S=728$.

To find a formula for the last term, we will write a progression having a for its first term. Then as each term is formed by multiplying the preceding term by r, the progression will be.

$$\begin{array}{cccccc} \text{1st.} & \text{2d.} & \text{3d.} & \text{4th.} & \text{5th.} & n\text{th.} \\ a, & ar, & ar^2, & ar^3, & ar^4, \; . \; . \; . & ar^{n-1} \end{array}$$

Note.—The dots are here used to supply the place of the intermediate terms, since n is indefinite (52).

It will be seen that the 2d term is formed by multiplying the 1st term by the ratio; the 3d term, by multiplying the 1st term by the second power of the ratio; the 4th term, by multiplying the 1st term by the third power of the ratio,—and so to find any term, multiply the first term by the ratio raised to a power whose index is less by 1 than the place of the given term. Thus the 10th term $= a \times r^9$, and the nth, or last term, will be $a \times r^{n-1}$. Hence,

$$l = ar^{n-1}.$$

That is,

219. *To find the last term in any geometrical progression, multiply the first term by the ratio raised to a power whose exponent is* 1 *less than the number of terms.*

Find the 6th term of the series 4, 12, 36, &c.

Here $a=4$, $r=3$, and $n=6$; then
$l = 4 \times 3^5 = 4 \times 243 = 962.$

1. Find the 8th term of the series 4374, 1458, 486, &c.

$$\text{Here } a=4374,\ r=\frac{1}{3},\ \text{and } n=8.$$

2. Find the 6th term of the series 9, 36, 144, &c.
3. Find the 5th term of the series 12, 3, $\frac{3}{4}$, &c.
4. Find the 12th term of the series 1, 2, 4, 8, &c.

The *sum* of the series is equal to the sum of all its terms; that is,

$$S=a+ar+ar^2+ar^3+\ldots+ar^{n-2}+ar^{n-1}.$$

Now multiplying this equation by r, we shall have

$$rS=ar+ar^2+ar^3+ar^4+\ldots+ar^{n-1}+ar^n.$$

Comparing the 2d equation with the 1st, it will be seen that all the terms in the 2d members of the two equations, except a and ar^n, are similar. Hence, if the 1st equation be subtracted from the 2d, we shall have

$$rS-S=ar^n-a.$$

$$S(r-1)=ar^n-a.$$

$$S=\frac{ar^n-a}{r-1}.$$

Since $l=ar^{n-1}$, multiplying by r we have

$$lr=ar^n.$$

Substituting lr for ar^n in the formula for S, we have

$$S=\frac{lr-a}{r-1}.$$

Factoring the first formula for S, we have

$$S=\frac{a(r^n-1)}{r-1}.$$

Translating these two formulas into words, we have the following rule for finding the sum of any geometrical progression.

220. (1.) *Multiply the last term by the ratio, subtract the first term from the product, and divide the remainder by* 1 *less than the ratio.*

(2.) *Raise the ratio to a power whose index shall be equal to the number of terms, subtract* 1 *from the result, multiply the remainder by the first term, and divide the product by* 1 *less than the ratio.*

Thus, to find the sum of 8 terms of the series 2, 6, 18, &c., by the first part of the rule, we must first find l.

$l = 2 \times 3^7 = 2 \times 2187 = 4374$, the last term.

$S = \frac{4374 \times 3 - 2}{3 - 1} = \frac{13122 - 2}{2} = 6560$, the sum of the series.

By the second part of the rule,

$S = \frac{a(r^n - 1)}{r - 1} = \frac{2(3^8 - 1)}{3 - 1} = \frac{2(6561 - 1)}{2} = 6560$, as above.

5. What is the sum of 6 terms of the series 9, 36, 144, &c.?

6. Find the 9th term and the sum of the first 9 terms of the series 3, 6, 12, &c.

7. Find the sum of 10 terms of the series 5, 15, 45, &c.

8. What is the 5th term, and the sum of the first 5 terms, of the series 2, 10, 50, &c.?

When the ratio is a fraction, as $\frac{1}{2}$, $\frac{1}{3}$, the equation $rS = ar + ar^2$, &c., produced by multiplying the sum of the first series by $r =$ the ratio, is *less* than the original series, and hence should be subtracted from it. Then we shall have, subtracting the second equation from the first,

$$S - rS = a - ar^n.$$
$$S(1 - r) = a - ar^n.$$
$$S = \frac{a - ar^n}{1 - r}.$$

This is the same as if we had changed all the signs in the value of S as first found. For if $S=\frac{ar^n-a}{r-1}$, and r be a fraction, it is evident that ar^n is less than a, and r less than 1.

Changing the signs in both numerator and denominator (which does not alter the value of the fraction), we shall have $S=\frac{-ar^n+a}{-r+1}=\frac{a-ar^n}{1-r}$, as above.

It will also be seen that where r is a fraction, the value of r^n decreases in proportion as n increases. That is, if $r=\frac{1}{2}$ and $n=5$; $r^n=\left(\frac{1}{2}\right)^5=\frac{1}{32}$. But if $n=10$, $r^n=\left(\frac{1}{2}\right)^{10}=\frac{1}{1024}$; if $n=20$, $\left(\frac{1}{2}\right)^{20}=\frac{1}{1,048,576}$; and if we suppose the series to be carried to infinity, in which case n becomes infinitely great, then the value of r^n becomes infinitely small, and may be regarded as zero.

If now we substitute 0 for r^n in the formula for the sum of a decreasing series, $S=\frac{a-ar^n}{1-r}$, we shall have

$$S=\frac{a-a.0}{1-r}, \text{ or } S=\frac{a}{1-r}.$$

This is the formula for the sum of an infinite decreasing series in geometrical progression, and may be thus translated into words:

221. *Divide the first term by the difference between the ratio and* 1.

To find the sum of 10 terms of the series 99, 33, 11, $\frac{11}{3}$, &c., we substitute in the formula $S=\frac{a(1-r^n)}{1-r}$, 99 for a, $\frac{1}{3}$ for r, and 10 for n. Then

$$S=\frac{99(1-(\frac{1}{3})^{10})}{1-\frac{1}{3}}.$$

$$\left(\frac{1}{3}\right)^{10}=\frac{1}{59049} \text{ and } 1-\left(\frac{1}{3}\right)^{10}=\frac{59048}{59049}\times 99\times\frac{3}{2}$$

$$=\frac{29524\times 11}{2187}.$$

$$S=\frac{324764}{2187}=148\tfrac{1088}{2187}, \text{ the sum of 10 terms of the series.}$$

To find the sum of the same series carried to infinity, we substitute in the formula $S=\dfrac{a}{1-r}$, 99 for a and $\dfrac{1}{3}$ for r.

$$S=\frac{99}{\frac{2}{3}}=99\times\frac{3}{2}=148\tfrac{1}{2}.$$

9. What is the sum of 5 terms of the series 12, 3, $\frac{3}{4}$, &c.?

10. Find the sum of the same series, carried to infinity.

11. Find the 10th term, and the sum of 10 terms of the series 1, $\frac{1}{10}$, $\frac{1}{100}$, &c.

12. Required the sum of the same series carried to infinity.

13. What is the sum of the progression 20, 4, $\frac{4}{5}$, &c., carried to infinity?

14. Find a mean proportional between 45 and 20.

15. There are three numbers in geometrical progression, of which the first term is 2, and the sum of the terms is 62. What are the numbers?

Let $x=$ the ratio; then the series will be
2, $2x$, and $2x^2$.

16. Required four numbers in geometrical progression, such that the sum of the first three shall be 147, and the sum of the last three 588.

Let $x=$ the first term, and y the ratio.
Then x, xy, xy^2, xy^3, will be the terms of the series.

NOTE.—Divide the 2d equation by the first.

17. There is an increasing geometrical progression consisting of three terms. The sum of the series is 325, and the difference between the first and the last term is 200. What are the numbers?

18. In a geometrical progression consisting of three terms, the sum of the first two terms is to the sum of the last two as 1 is to 5, and the sum of the series is 93. Required the terms of the series.

CHAPTER X.

MISCELLANEOUS EXAMPLES.

1. Divide $9289 between A and B, in such a manner that A's share shall be to B's as 2 to 5.

2. A man, dying, left an estate valued at $14832. In his will he gave $\frac{1}{2}$ of his property to his wife; and directed the remainder to be so divided between his son and daughter, that the daughter's portion might be to the son's as 3 to 5. What was the share of each?

3. What is that number, to which if you add $\frac{1}{2}$ of itself, and from the result subtract $\frac{3}{4}$ of the sum, $\frac{1}{3}$ of the remainder is 3?

4. A farmer bought 12 sheep, 5 cows, 2 yoke of oxen, and 3 horses, for 795 dollars. A cow cost as much as 6 sheep, a yoke of oxen as much as 3 cows, and a horse as much as 3 oxen. What did he give for each?

5. A man divided a certain sum of money equally between his son and daughter; but had he given his son 33 dollars more, and his daughter 47 dollars less, her share would have been but $\frac{1}{3}$ of his. What was the sum divided?

6. Divide 46 dollars into two such parts that $\frac{1}{7}$ of one and $\frac{1}{3}$ of the other may be 10 dollars.

7. A man divided 198 acres of land between his three children, in such a manner that A's part was to B's in the ratio of 3 to 8; and C had as many acres as both his brothers. What was the share of each?

8. A man bought a certain quantity of wine for 94 dollars; and after 7 gallons had leaked out, he sold $\frac{1}{4}$ of the remainder, at cost, for \$20. How many gallons did he buy?

9. If a certain number be divided by the sum of its digits, the quotient will be 8; but if the digits be inverted, and that number divided by 2 less than their difference, the quotient will be 9. What is the number?

10. Two friends bought a horse together; and when one had paid $\frac{2}{5}$ and the other $\frac{4}{9}$ of the price agreed upon, they still owed 21 dollars. What was the price of the horse?

11. In a certain university there are 384 students, $\frac{3}{4}$ of whom belong to the academical department; and in the departments of law, divinity, and medicine, the students are to each other as the numbers 1, 2, and 3. How many students are there in each department?

12. Divide \$1170 among three persons, A, B, and C, in proportion to their ages. Now, B is a third part older than A, and A is half as old as C. What is the share of each?

13. Three men, A, B, and C, pay a tax of 594 dollars. The property of A is to that of B as 3 to 5; and the property of B is to that of C as 8 to 7. What part of the tax is paid by each?

14. A father gives to his six sons \$2010, which they are to divide according to their ages, so that each elder son shall receive \$24 more than his next younger brother. What is the share of the youngest son?

15. If I multiply a certain number by 6, add 18 to the product, and divide the sum by 9, the quotient will be 20. What is the number?

16. Divide 119 into three such parts that the second will be 3 more than 3 times the first, and the third 3 more than 3 times the second.

17. A schoolmaster, being asked how many dollars he received a month for teaching, replied, "If I add 9 to $\frac{1}{4}$ part of the number of dollars I receive, multiply the sum thus obtained by 7, subtract 15 from the product, multiply the remainder by 6, and then take away the cipher from the right of the number last obtained, I shall have $54." What were his wages?

18. Some travellers find a purse of money, which they agree to share equally. If they take 5 dollars apiece, one man will receive nothing; but if they take 4 dollars, there will be 7 dollars left. What is the number of travellers? What is the sum to be divided?

19. There are two numbers, the product of whose sum, multiplied by the greater, is 144; and whose difference, multiplied by the less, gives 14. Required the numbers.

20. A courier had been travelling 4 days, at the rate of 6 miles an hour, when another was sent after him, who travelled 8 miles an hour. In how many days will the second courier overtake the first, if they both travel 15 hours a day?

21. A gentleman has a rectangular garden whose perimeter is 36 rods; and the square of the width is to the square of the length as 16 to 25. Required its dimensions.

22. Two travellers, A and B, began a journey of 300 miles at the same time. A travelled a mile an hour faster than B, and arrived at his journey's end 10 hours before him. How many miles an hour did each travel?

23. Two sportsmen, walking over a marsh, started a flock

of plover. The first one fired, and brought down $\frac{8}{9}$ of the whole flock. Afterwards, the second one fired, and killed a number equal to the square root of half the flock; when only 2 birds were left. How many birds were there in the flock?

NOTE.—Let $x =$ the number killed by the second shot.

24. Required the side of a square field, which shall contain the same quantity of land as another field, which is 72 rods long and 18 rods wide.

25. Three planters, A, B, and C, together possess 2658 acres of land. If B sell A 215 acres, then will A's plantation exceed B's by 236 acres; but if B buy 167½ acres of C's plantation, both will have the same quantity of land. How many acres has each?

26. Required two numbers, whose sum, multiplied by their product, shall be equal to 12 times the difference of their squares; the numbers being to each other in the ratio of 2 to 3.

27. It is required to form a regiment, containing 865 men, into two squares, one of which shall contain 7 more men in rank and file than the other. How many men must each of the squares contain?

28. A man, having travelled 108 miles, found that he could have performed the same journey in 6 hours less, if he had travelled 3 more miles an hour. At what rate did he travel?

29. Two persons, A and B, set out at the same time from two towns, distant 396 miles; and, having travelled as many days as A travelled miles daily more than B, they met each other. It then appeared that A had travelled 216 miles. How many miles did each travel per day?

30. Two merchants, A and B, trade in company, and gain \$1930.28. Of the capital employed, A furnished \$4000 and B \$7000. What is each man's share of the gain?

31. A farmer being asked how many acres of land he

owned, answered, that the number was expressed by two digits, whose sum, increased by 7, would be equal to three times the left-hand digit; and he added, that, if he owned 18 acres less, the digits expressing the number would be inverted. How many acres were there in his farm?

32. Several gentlemen made an excursion, each taking the same sum of money. Each had as many servants attending him as there were gentlemen; the number of dollars which each had was double the number of all the servants, and the whole sum of money taken out was 3456 dollars. How many gentlemen were there?

33. Four farmers, A, B, C, and D, hired a pasture, for which they paid 81 dollars. A put in 4 cows for 3 months; B, 8 cows for 2 months; C, 7 cows for 5 months; and D, 3 cows for 6 months. How much of the rent must each man pay?

34. There is a certain number, the left-hand digit of which is equal to 3 times the right-hand digit; and if 12 be subtracted from the number, the remainder will be equal to the square of the left-hand digit. Required the number.

35. A man has two horses and two saddles, one of which is worth $40, and the other $5. When the best saddle is upon the first horse, and the worst saddle upon the second, the former is worth just twice as much as the latter; but when the worst saddle is upon the first horse, and the best saddle upon the second, the latter is worth $5 more than the former. What is the value of each horse?

36. Find three such numbers, that the first, with $\frac{1}{2}$ the sum of the second and third, shall be 78; the second, with $\frac{1}{3}$ the excess of the third over the first, shall be 60; and $\frac{1}{2}$ the sum of the three shall be 66.

37. If I had 3 shillings more in my pocket, I could give 2*s.* 6*d.* to each of a certain number of beggars; but if I give them

only 2*s.* apiece, I shall have 4*s.* left. How much money have I in my pocket? What is the number of beggars?

38. A person had £27 6*s.* in guineas and crown pieces. Having paid a debt of £14 17*s.*, he finds that he has as many guineas left as he has paid away crowns; and as many crowns left as he has paid away guineas. How many crowns and guineas had he at first?

REMARK.—A guinea is 21 shillings, and a crown 5 shillings, sterling.

39. A laborer agreed to work 24 days for 75 cents a day, and to forfeit his wages and 25 cents every day he was idle. At the end of the time, he received $12. How many days was he idle?

40. Generalize the preceding example, by letting n equal the number of days he agreed to labor, receiving a shillings for every day he worked, and forfeiting b shillings for every day he was idle. If he received d shillings at the end of the time, how many days was he idle?

41. A cistern, containing 276 gallons, is emptied in 21 minutes by two cocks running successively. One discharges 16 gallons, and the other 11 gallons, in a minute. How many minutes is each running?

42. A merchant has two kinds of wine; one of which is worth 9*s.* 6*d.* per gallon, and the other, 13*s.* 6*d.* How many gallons of each must he take, to form a mixture of 104 gallons which shall be worth £56?

43. A gentleman bought a quantity of broadcloth for $48; and four times the number of yards were equal to three times the price of a yard. How many yards did he buy, and at what price?

44. Two gentlemen, A and B, have rectangular gardens contiguous to each other. A's garden is 20 yards wide, and

$\frac{2}{3}$ as long as B's; and the surface of B's garden is to that of A's as 5 to 3. What is the width of B's garden?

45. A miser, dying, left a certain number of eagles, as many quarter-eagles, $\frac{3}{8}$ the number of half-eagles, and dollars enough to make the whole number of coins equal to $\frac{1}{5}$ of the value of the whole in dollars; and the eagles and dollars together were 2 more than $\frac{1}{2}$ the number of coins. How much money did he leave?

46. A farmer sold 120 bushels of rye and barley; receiving, for a bushel of each kind of grain, as many cents as there were bushels of that kind; and the barley brought only $\frac{4}{9}$ as much as the rye. How many bushels of each kind did he sell?

47. A gentleman distributed $47.50 among 30 men and women, giving the women 8*s.* and the men 10*s.* 6*d.* each. How many men and how many women were there?

48. A criminal, having escaped from prison, travelled 10 hours before his escape was known. He was then pursued, so as to be gained upon 3 miles an hour. After his pursuers had travelled 8 hours, they met an express going at the same rate as themselves, who met the criminal 2 hours and 24 minutes before. In what time from the commencement of the pursuit did they overtake him?

49. A farmer has an irregular piece of land, containing 5 acres, which he wishes to exchange for a square field of the same size. Required one of the sides of the square field.

REMARK.—An acre of land contains 160 square rods. Only an *approximate* answer to this question can be found, as the given quantity is not a perfect square.

50. I have a field containing 10 acres; and the length of the field exceeds its width by 18 rods. Required its dimensions.

51. A man bought a field whose length was to its breadth as 8 to 5. The number of dollars paid per acre was equal to the number of rods in the length of the field; and the number of dollars paid for the whole was equal to 13 times the number of rods round the field. What did he give for the field?

52. A father gave to each of his children, on new year's day, as many books as he had children; for each book he gave 12 times as many cents as there were children; and the cost of the whole was $15. How many children had he?

53. A messenger had been gone from a certain place 8 hours, when another was sent after him. The first went 7 miles an hour, and the second 11. In what time did the second overtake the first?

54. A man wished to plant a certain number of trees in the form of a square. At the first trial, he had 39 trees left. He then determined to enlarge the square by adding one tree to each row; to do which, he found it necessary to procure 50 trees more. How many trees had he at first?

55. A grocer, being asked the size of 3 wine-casks, replied, "If I fill the first cask from the second, $\frac{2}{9}$ of the wine will remain; if I fill the second from the third, $\frac{1}{8}$ of the wine will remain; and the third cask will contain the contents of the first cask and 23 gallons more." Required the size of the casks.

56. A cistern, which holds 2340 gallons, is filled in $\frac{3}{4}$ of an hour by 3 pipes; the first of which conveys 13 gallons more, and the second 6 gallons less, than the third per minute. How many gallons does each pipe convey in a minute?

57. Two persons, A and B, set out at the same time from two towns at a distance of 672 miles. B travelled 8 miles a day more than A; and when they had travelled half as many

days as A went miles in a day, they met. How many miles did each travel daily?

58. A farmer has a rectangular peach-orchard, with unequal sides. If the difference of the sides be multiplied by the greater side, and the product divided by the less, the quotient is 24 rods; but if their difference be multiplied by the less side, and the product divided by the greater, the quotient is only 6 rods. What are the dimensions of the orchard?

59. There is a school-room in Boston, whose length is to its breadth as 6 to 5. If it were a square, having its sides equal to the length, it would contain 891 feet more than it would were the sides of the square equal to its width. What are the dimensions of the room?

60. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel be increased one yard, it will make only 4 revolutions more in going the same distance. What is the circumference of each wheel?

61. Find a mean proportional between $\frac{1}{2}$ and 15.

62. If 20 be added to the square root of a certain number, the cube root of the sum will be 3. Required the number.

63. Find two numbers, the sum of whose second powers is 61, and the second power of whose sum is 121.

64. Divide 104 into three numbers in geometrical progression; the terms to be such that the sum of the first and second shall be to the sum of the first and third, as 2 is to 5.

65. In a geometrical progression, consisting of four terms, the sum of the first three terms is 52, and the sum of the last three is 156. What are the terms?

66. There is a number, consisting of three digits, to which if 396 be added, the digits will be inverted. Moreover, the sum

of the squares of the digits is 56, and the square of the middle digit exceeds by 4 the product of the other two digits. What is the number?

67. Two travellers, 500 miles apart, set out to meet each other. One travels 6 miles the first day, 8 the second, 10 the third, and so on till they meet. The other starts 5 days after the first, and travels regularly 20 miles a day. In how many days will they meet, and how many miles will each travel?

68. If the sum of two numbers be increased by 4, and then the square root of the sum be taken, the result added to the sum of the numbers will be 68. Moreover, the difference of the squares of the two numbers is 240. What are the numbers?

69. There is a certain fraction, such that if we extract the square root of the numerator, and double the denominator, the value of the fraction becomes $\frac{1}{5}$. Also, if we add 1 to the square root of the numerator, and subtract 1 from the denominator, the value of the fraction becomes $\frac{3}{4}$. What is the fraction?

70. If the square of a certain number be taken from 40, and the square root of the difference be increased by 10, and the sum be multiplied by 2, and the product divided by the number itself, the quotient will be 4. Required the number.

71. Two clerks, A and B, sent ventures in a ship bound to India. A gained $11; and, at this rate, he would have gained as many dollars on a hundred as B sent out. B gained $36, which was but one-fourth part as much per cent as A gained. How much money was sent out by each?

72. Required two fractions, whose product is $\frac{1}{9}$, and the sum of whose squares is $\frac{17}{36}$.

73. A company of persons spend £3 10*s*. at a tavern. Four of them go away without paying; in consequence of which each

of the others has to pay 2*s.* more than his proper share. How many persons were there in the company? and what was the proper share of each?

74. A gentleman bought a rectangular lot of land, giving $10 for every foot in the perimeter. If the same quantity of land had been in the form of a square, and he had bought it in the same way, it would not have cost him so much by $330; and if he had bought a square piece of the same perimeter, he would have had $12\frac{1}{4}$ rods more. What were the dimensions of the lot?

[illegible]
These letters [illegible]

to "Kelvy" these words are
respectfully dedicated —
by H. G. Olney
M. J. [illegible]

THE END.

[illegible] in this you [illegible]
me